CMPUT 267 Basics of Machine Learning

Recap

This class is about **understanding** machine learning techniques by understanding their basic **mathematical underpinnings**

- Please read FAQ document on course webpage.
- Course information at https://nidhihegde.github.io/mlbasics
- eClass: https://eclass.srv.ualberta.ca/course/view.php?id=95783
- Readings from online https://marthawhite.github.io/mlbasics/notes.pdf
- Assignment 1 will be released by the end of the week.
- First participation and readings question-exercise will be released next Tuesday.

Probability Theory

CMPUT 267: Basics of Machine Learning

Today’s slide contents re-used from Martha White and James Wright

Outline

1. Probabilities
2. Defining Distributions
3. Random Variables

Why Probabilities?

Even if the world is completely deterministic, outcomes can **look random** (**why?**)

Example: A high-tech gumball machine behaves according to
$f(x_1, x_2) =$ output candy if x_1 & x_2,
where x_1 = **has candy** and x_2 = **battery charged**.

- You can only see if it has candy
- From your perspective, when $x_1 = 1$, sometimes candy is output, sometimes it isn't
- It **looks stochastic**, because it depends on the hidden input x_2

Measuring Uncertainty

- **Probability** is a way of **measuring** uncertainty
- We assign a number between 0 and 1 to **events** (hypotheses):
  - **0** means absolutely certain that statement is **false**
  - **1** means absolutely certain that statement is **true**
  - **Intermediate** values mean more or less certain
- Probability is a measurement of **uncertainty**, **not truth**
  - A statement with probability .75 is not "mostly true"
  - Rather, we **believe** it is more **likely** to be true than not

Example

- Let’s think about estimating the average height of a person in the world
- There is a true population mean h (say h = 165.2 cm)
  - which can be computed by averaging the heights of every person
- We can estimate this true mean using data
  - e.g., compute a sample average $\bar{h}$ from a subpopulation by randomly sampling 1000 people from around the whole world (say $\bar{h}$ = 166.3 cm)
- We can also reason about our belief over plausible estimates $\bar{h}$ of h
  - e.g., we can maintain a distribution over plausible $\bar{h}$, such as saying $p(\bar{h} = 160) = 0.1$, $p(\bar{h} = 163) = 0.3$, $p(\bar{h} = 165) = 0.5$, $p(\bar{h} = 167) = 0.1$

Prerequisites Check

- Derivatives
  - Rarely integration
  - Partial derivatives
- Vectors, dot-products, matrices
- Set notation
  - Complement A^c of a set, union $A \cup B$ of sets, intersection of sets $A \cap B$
  - Set of sets, power set $\mathscr{P}(A)$
- Basics of probability. (We will refresh today)

Terminology Refresher

- If you are unsure, notation sheet in the notes is a good starting point
- Set notation
  - Curly brackets for discrete sets, e.g $\{a, b, c\}$, $\{1,2,3,4,5\}$, $\{-2.1,6.5\}$
  - Square brackets for continuous intervals, e.g., $[-10,10]$, $[3.2,7.1]$
  - Subset notation $A \subset \Omega$ and the set complement $A^c = \Omega \backslash A$
  - Union of sets $A \cup B$, intersection of sets $A \cap B$
  - Power set $\mathscr{P}(A)$, e.g, $A = \{1,2\}$, $\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
- Scalar $x \in \mathbb{R}$ and vector (array) is $\mathbf{x} \in \mathbb{R}^d$ for some integer $d \in \{2,3,\ldots\}$

Terminology - cont'd

- **Countable:** A set whose elements can be assigned an integer index
  - The integers themselves
  - Any finite set, e.g., $\{0.1, 2.0, 3.7, 4.123\}$
  - We'll sometimes say **discrete**, even though that's a little imprecise
- **Uncountable:** Sets whose elements *cannot* be assigned an integer index
  - Real numbers $\mathbb{R}$
  - Intervals of real numbers, e.g., $[0,1]$, $(-\infty, 0)$
  - Sometimes we'll say **continuous**

Outcomes and Events

All probabilities are defined with respect to a **measurable space** $(\Omega, \mathscr{E})$ of **outcomes** and **events**:

- Ω is the **sample space**: The set of all possible outcomes
- $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is the **event space**: A set of subsets of Ω satisfying two key properties

Examples of Discrete & Continuous Sample Spaces and Events

Discrete (countable) outcomes

$\Omega = \{1,2,3,4,5,6\}$

$\Omega = \{\text{person, robot, camera, TV}, \ldots\}$

$\Omega = \mathbb{N}$

Continuous (uncountable) outcomes

$\Omega = [0,1]$

$\Omega = \mathbb{R}$

$\Omega = \mathbb{R}^k$

Outcomes and Events

All probabilities are defined with respect to a **measurable space** $(\Omega, \mathscr{E})$ of **outcomes** and **events**:

- Ω is the **sample space**: The set of all possible outcomes
- $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is the **event space**: A set of subsets of Ω satisfying
  1. $A \in \mathscr{E} \implies A^c \in \mathscr{E}$
  2. $A_1, A_2, \ldots \in \mathscr{E} \implies \bigcup_{i=1}^{\infty} A_i \in \mathscr{E}$

Event Spaces

Definition:
A set $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is an **event space** if it satisfies

1. $A \in \mathscr{E} \implies A^c \in \mathscr{E}$
2. $A_1, A_2, \ldots \in \mathscr{E} \implies \bigcup_{i=1}^{\infty} A_i \in \mathscr{E}$

1. A collection of outcomes (e.g., either a 2 or a 6 were rolled) is an event.
2. If we can measure that an event has occurred, then we should also be able to measure that the event has not occurred; i.e., its **complement** is measurable.
3. If we can measure two events separately, then we should be able to tell if one of them has happened; i.e., their **union** should be measurable too.

Discrete vs. Continuous Sample Spaces

Discrete (countable) outcomes

$\Omega = \{1,2,3,4,5,6\}$

$\Omega = \{\text{person}, \text{woman}, \text{man}, \text{camera}, \text{TV}, \ldots\}$

$\Omega = \mathbb{N}$

$\mathscr{E} = \{\emptyset, \{1,2\}, \{3,4,5,6\}, \{1,2,3,4,5,6\}\}$

Typically**:** $\mathscr{E} = \mathscr{P}(\Omega)$

Question:
$\mathscr{E} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$?

Continuous (uncountable) outcomes

$\Omega = [0,1]$

$\Omega = \mathbb{R}$

$\Omega = \mathbb{R}^k$

$\mathscr{E} = \{\emptyset, [0,0.5], (0.5,1.0], [0,1]\}$

Typically: $\mathscr{E} = B(\Omega)$ ("Borel field")

Note: *not* $\mathscr{P}(\Omega)$

Exercise

- Write down the power set of {1, 2, 3}
- More advanced: Why is the power set a valid event space? Hint: Check the two properties

Definition:

A non-empty set $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is an **event space** if it satisfies

1. $A \in \mathscr{E} \implies A^c \in \mathscr{E}$
2. $A_1, A_2, \ldots \in \mathscr{E} \implies \bigcup_{i=1}^{\infty} A_i \in \mathscr{E}$

Exercise answer

A set $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is an **event space** if it satisfies

1. $A \in \mathscr{E} \implies A^c \in \mathscr{E}$
2. $A_1, A_2, \ldots \in \mathscr{E} \implies \bigcup_{i=1}^{\infty} A_i \in \mathscr{E}$

- $\Omega = \{1,2,3\}$
- $\mathscr{P}(\Omega) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- Proof that the power set satisfies the two properties
- Take any $A \in \mathscr{P}(\Omega)$ (e.g., $A = \{1\}$ or $A = \{1,2\}$). Then $A^c = \Omega \backslash A$ is a subset of Ω, and so $A^c \in \mathscr{P}(\Omega)$ since the power set contains all subsets
- Take any $A, B \in \mathscr{P}(\Omega)$. Then $A \cup B \subset \Omega$, and so $A \cup B \in \mathscr{P}(\Omega)$
- More generally, for an infinite union, see: https://proofwiki.org/wiki/Power_Set_is_Closed_under_Countable_Unions

Axioms

Definition:

Given a measurable space $(\Omega, \mathscr{E})$, any function $P : \mathscr{E} \to [0,1]$ satisfying

1. **unit measure:** $P(\Omega) = 1$, and
2. **σ-additivity:** $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ for any countable sequence $A_1, A_2, \ldots \in \mathscr{E}$ where $A_i \cap A_j = \varnothing$ whenever $i \neq j$

is a **probability measure** (or **probability distribution**).

If P is a probability measure over $(\Omega, \mathscr{E})$, then $(\Omega, \mathscr{E}, P)$ is a **probability space**.

Defining a Distribution

Example:

$$\Omega = \{0,1\}$$

$$\mathscr{E} = \{\emptyset, \{0\}, \{1\}, \Omega\}$$

$$P = \begin{cases} 1 - \alpha & \text{if } A = \{0\} \\ \alpha & \text{if } A = \{1\} \\ 0 & \text{if } A = \emptyset \\ 1 & \text{if } A = \Omega \end{cases}$$

where $\alpha \in [0,1]$.

Questions:

1. Do you recognize this distribution?
2. How should we choose P in practice?
   a. Can we choose an arbitrary function?
   b. How can we guarantee that all of the constraints will be satisfied?

We will define distributions using **PMFs** and **PDFs**

PMF: probability mass function

PDF: probability density function

Probability Mass Functions (PMFs)

Definition: Given a **discrete** sample space Ω and event space $\mathscr{E} = \mathscr{P}(\Omega)$, any function $p : \Omega \to [0,1]$ satisfying $\sum_{\omega \in \Omega} p(\omega) = 1$ is a **probability mass function**.

- For a discrete sample space, instead of defining P directly, we can define a **probability mass function** $p : \Omega \to [0,1]$.
- p gives a probability for **outcomes** instead of **events**
- The probability for any event $A \in \mathscr{E}$ is then defined as $P(A) = \sum_{\omega \in \Omega} p(\omega)$.

Example: PMF for a Fair Die

A **categorical distribution** is a distribution over a **finite** outcome space, where the probability of each outcome is specified separately.

Example: Fair Die

$$\Omega = \{1,2,3,4,5,6\}$$

$$p(\omega) = \frac{1}{6}$$

ω	$p(\omega)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Questions:

1. What is a possible event? What is its probability?
2. What is the event space?

Moving to Boolean Terminology with Random Variables

Example: Suppose we observe both a die's number, and where it lands.

$$\Omega = \{(left, 1), (right, 1), (left, 2), (right, 2), \ldots, (right, 6)\}$$

We might want to think about the probability that we get a large number, without thinking about where it landed.

$$P(\{\omega \in \Omega \mid \omega_2 = 3\})$$

Let X = number that comes up. We could ask about $P(X = 3)$ or $P(X \geq 4)$

This notation is simpler to write than using the event notation above

$P(X = 3)$ would be written instead of $P(\{\omega \in \Omega \mid \omega_2 = 3\})$

Random Variables, Formally

Given a probability space $(\Omega, \mathscr{E}, P)$, a **random variable** is a function $X : \Omega \to \mathscr{X}$ (where $\mathscr{X}$ is a new outcome space), satisfying

$$\{\omega \in \Omega \mid X(\omega) \in A\} \in \mathscr{E} \quad \forall A \in B(\mathscr{X}).$$

It follows that $P_X(A) = P(\{\omega \in \Omega \mid X(\omega) \in A\})$.

Example: Let Ω be a population of people, $\omega =$ (height, age, $\ldots$, location), and $X(\omega)$ = height in cm, and the event $A = [150,170]$.

$$P(X \in A) = P(150 \le X \le 170) = P(\{\omega \in \Omega : X(\omega) \in A\}).$$

RVs are intuitive

- All the probability rules remain the same, since RVs are a mapping to create a new outcome space, event space and probabilities
- The notation may look onerous, but they simply formalize something we do naturally: specify the variable we care about, knowing it is defined by a more complex underlying distribution
- We have really already been talking about RVs
  - e.g., for $X =$ dice outcome, event $A = \{5,6\}$, $P(A) = P(X \geq 4)$

Random Variables and Events

- A Boolean expression involving random variables defines an event:

  E.g., $P(X \geq 4) = P(\{\omega \in \Omega \mid X(\omega) \geq 4\})$

- Similarly, every event can be understood as a Boolean random variable:

$$Y = \begin{cases} 1 & \text{if event } A \text{ occurred} \\ 0 & \text{otherwise.} \end{cases}$$

- From this point onwards, we will exclusively reason in terms of random variables rather than probability spaces.

Revisiting the Fair Die PMF

Example: Fair Die

$\mathcal{X} = \{1,2,3,4,5,6\}$

$p(x) = \frac{1}{6}$

x	$p(x)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Questions:

1. What is a possible event?
   What is its probability?

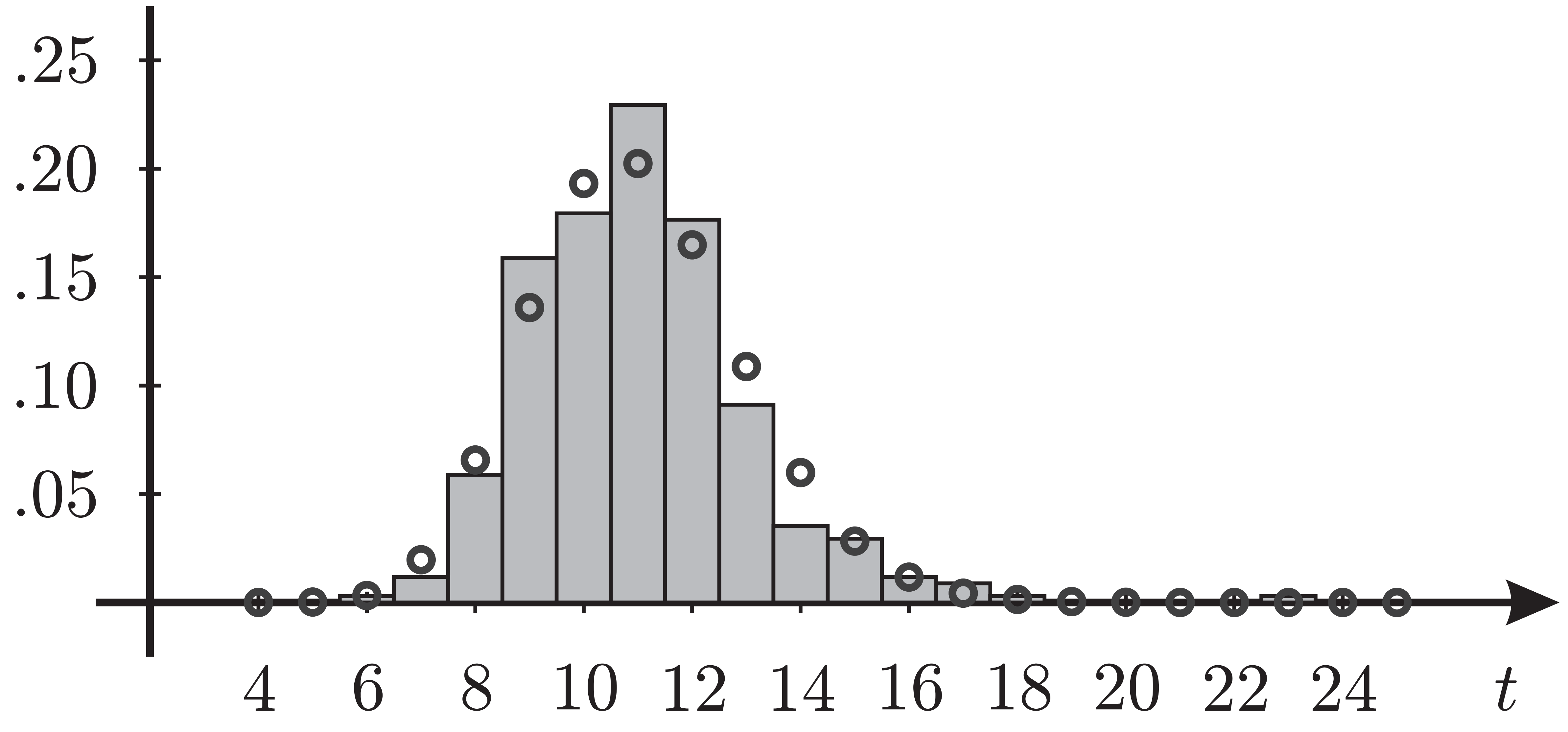
2. What is the event space?

Answer: event space and probabilities are the same, but we write the probabilistic question using booleans

$p(\{3,4\}) = \frac{1}{3} \longrightarrow p(3 \leq X \leq 4) = \frac{1}{3}$ or $p(X \in \{3,4\}) = \frac{1}{3}$

Example: Using a PMF

- Suppose that you recorded your commute time (in minutes) every day for a year (i.e., 365 recorded times).
- The random variable is T with outcomes $t \in \{4,5,6,7,\ldots,25\}$
- **Question:** How do you get $p(t)$?
- **Question:** How is $p(t)$ useful?
- **Question:** How do you compute $p(10 \leq T \leq 13)$?



Example: Using a PMF

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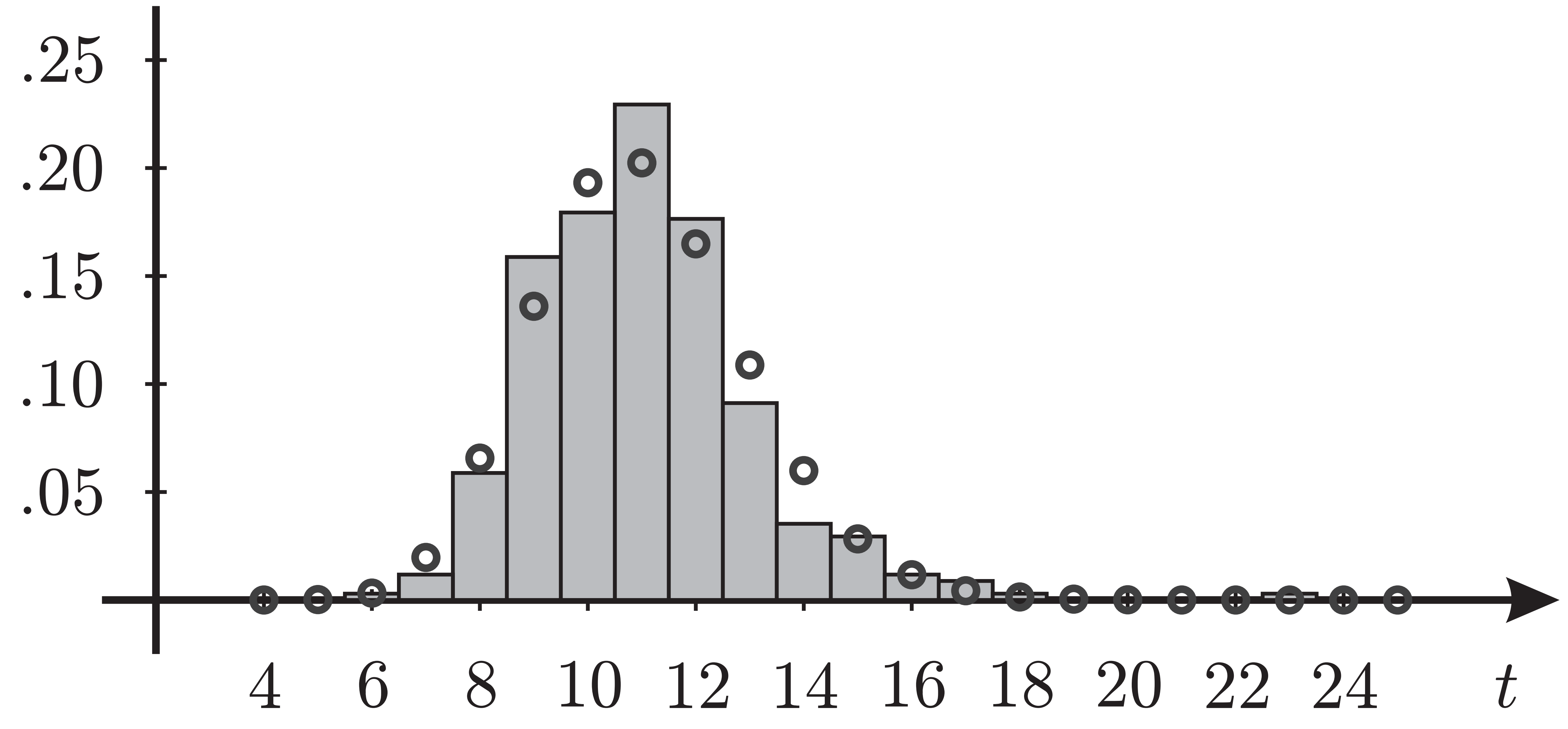
- The random variable is T with outcomes $t \in \{4,5,6,7,\ldots,25\}$
- **Question:** How do you get $p(t)$? (Answer: count and normalize)
- **Question:** How is $p(t)$ useful?
  - We can take mode as prediction
- **Question:** How do you compute $p(10 \leq T \leq 13)$?
  - Answer: $\displaystyle\sum_{t \in \{10,11,12\}} p(t)$



This PMF is called a categorical distribution, with 21 categories (table of probabilities)

Useful PMFs: Bernoulli

A **Bernoulli distribution** is a special case of a **categorical distribution** in which there are only two outcomes. It has a single **parameter** $\alpha \in (0,1)$.

$\Omega = \{T, F\}$, $\Omega = \{H, T\}$

$$p(\omega) = \begin{cases} \alpha & \text{if } \omega = T \\ 1 - \alpha & \text{if } \omega = F \,. \end{cases}$$

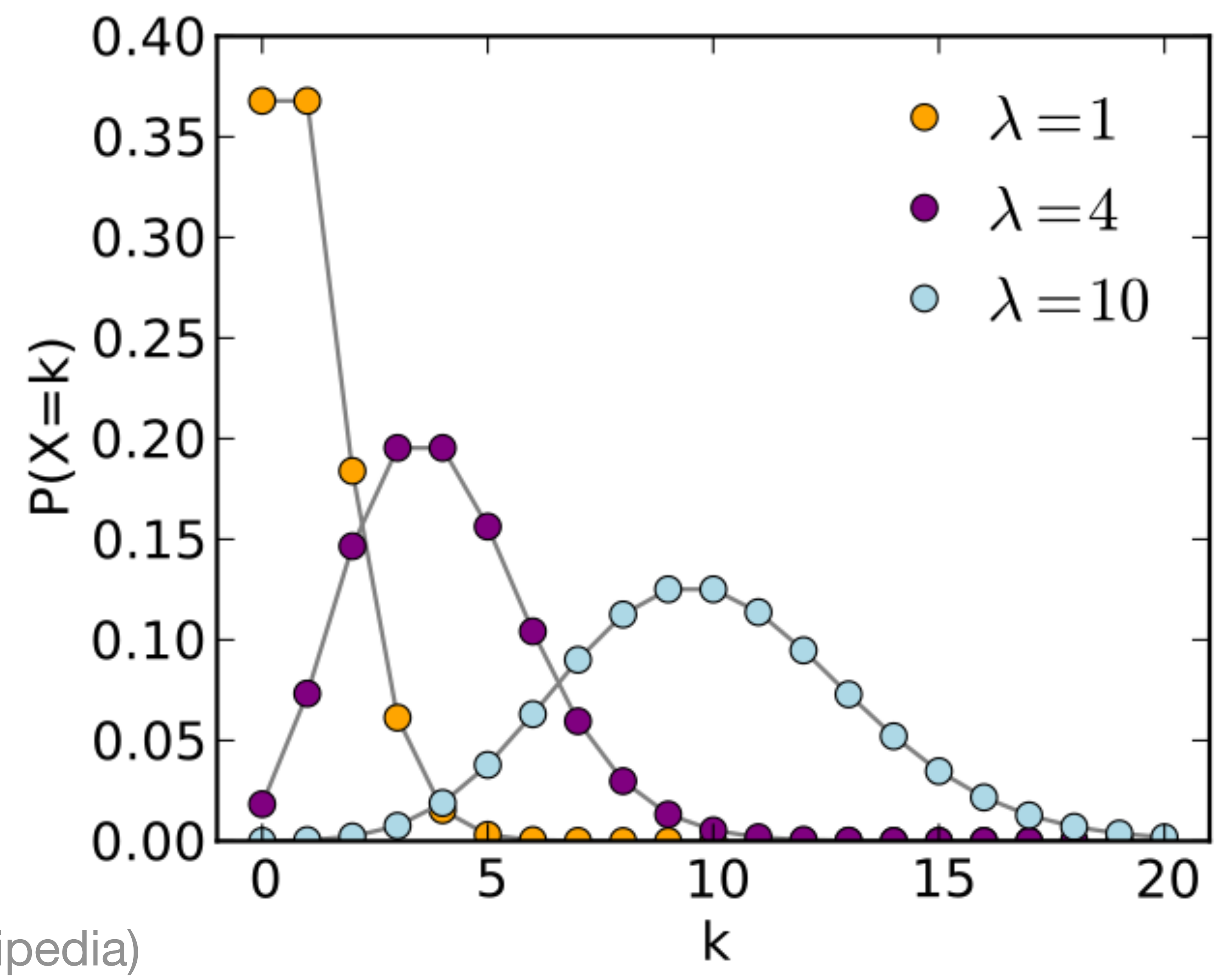
Alternatively: $\Omega = \{0,1\}$

$p(k) = \alpha^k (1-\alpha)^{1-k}$ for $k \in \{0,1\}$

Useful PMFs: Poisson

A **Poisson distribution** is a distribution over the non-negative integers.
It has a single parameter $\lambda \in (0,\infty)$.

E.g., number of calls received by a call centre in an hour, number of letters received per day.



(Image: Wikipedia)

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Questions:

1. Could we define this with a table instead of an equation?
2. How can we check whether this is a valid PMF?
3. λ real-valued, but outcomes are discrete. What might be the mode (most likely outcome)?

Useful PMFs: Poisson

A **Poisson distribution** is a distribution over the non-negative integers.
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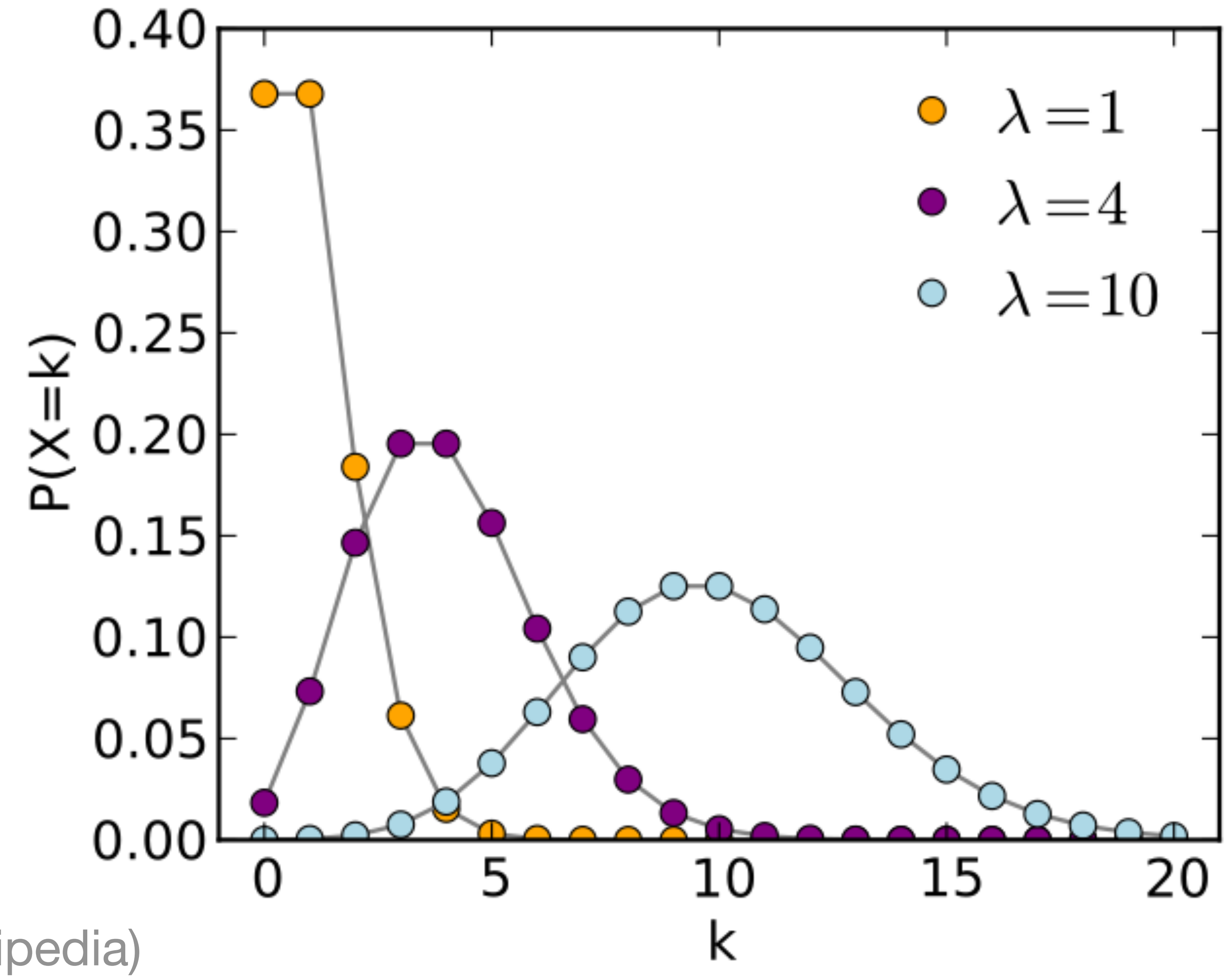


(Image: Wikipedia)

1. Could we define this with a table instead of an equation?
   - No because the outcome space is infinite

2. How can we check whether this is a valid PMF?
   - Check if $\sum_{k=0}^{\infty} p(k) = 1$

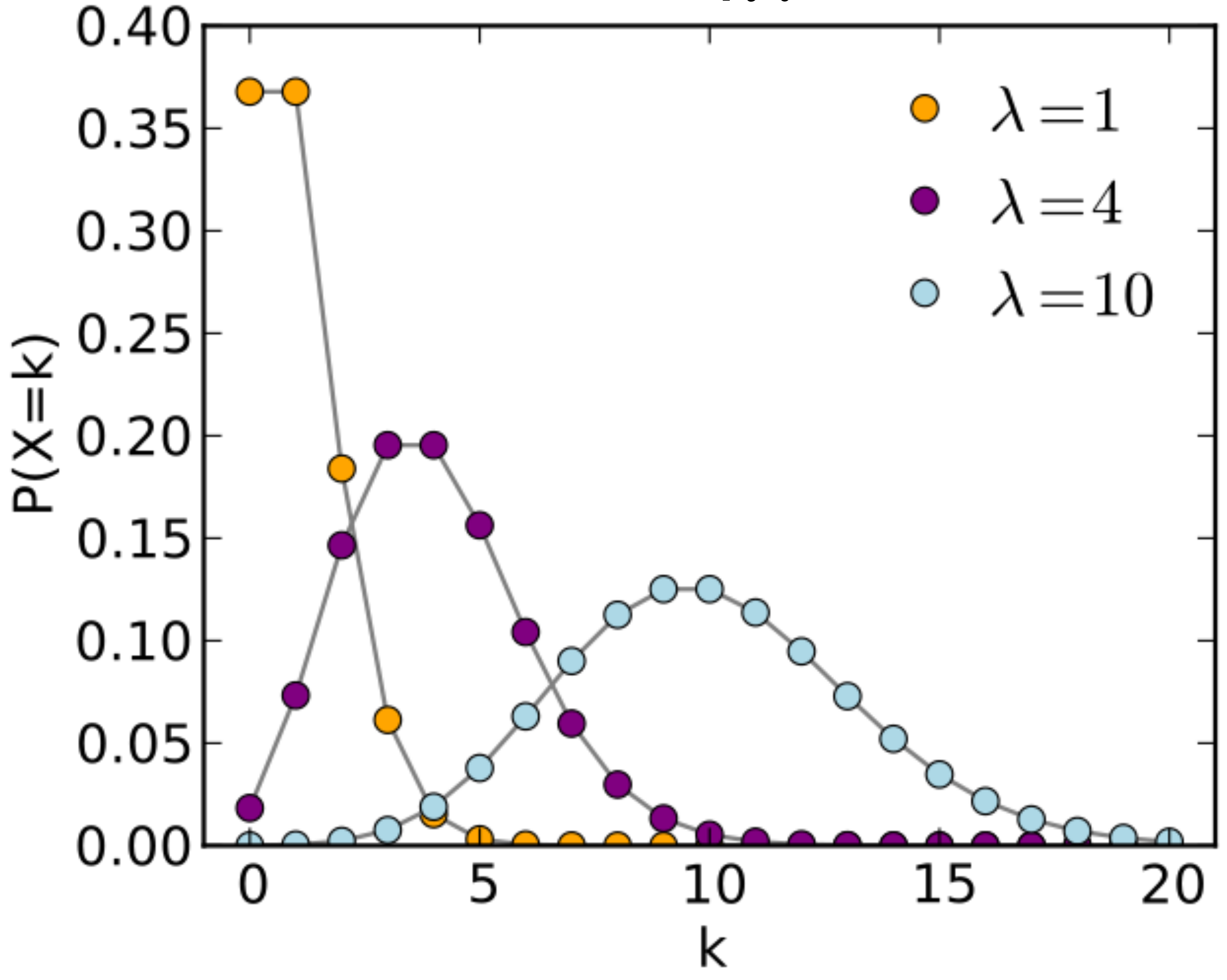
3. λ real-valued, but outcomes are discrete. What might be the mode (most likely outcome)?
   - Mean is λ, may not correspond to any outcome
   - Two modes, $\lceil \lambda \rceil - 1, \lfloor \lambda \rfloor$

Commute Times Again

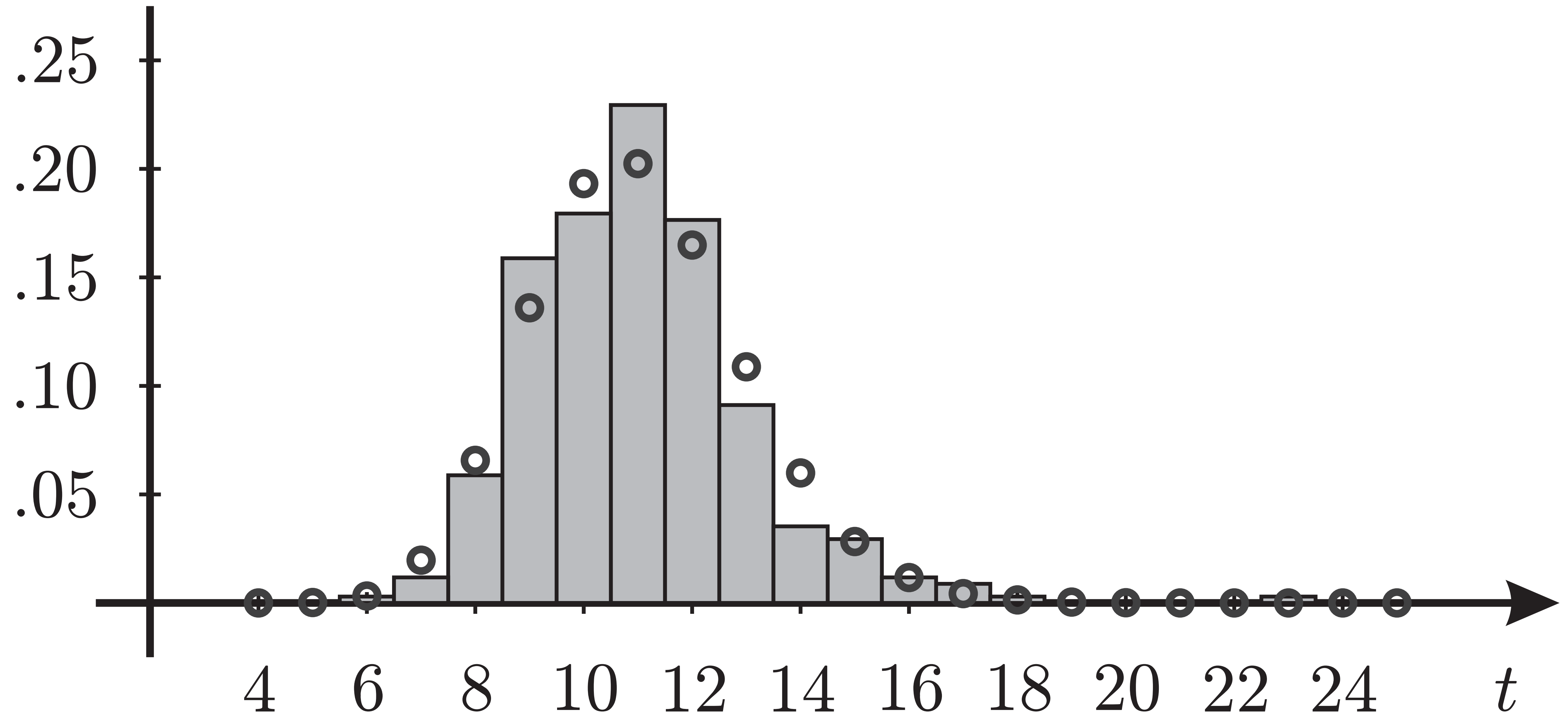
- **Question:** Could we use a **Poisson distribution** for commute times (instead of a categorical distribution)?

- **Question:** What would be the benefit of using a Poisson distribution?

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

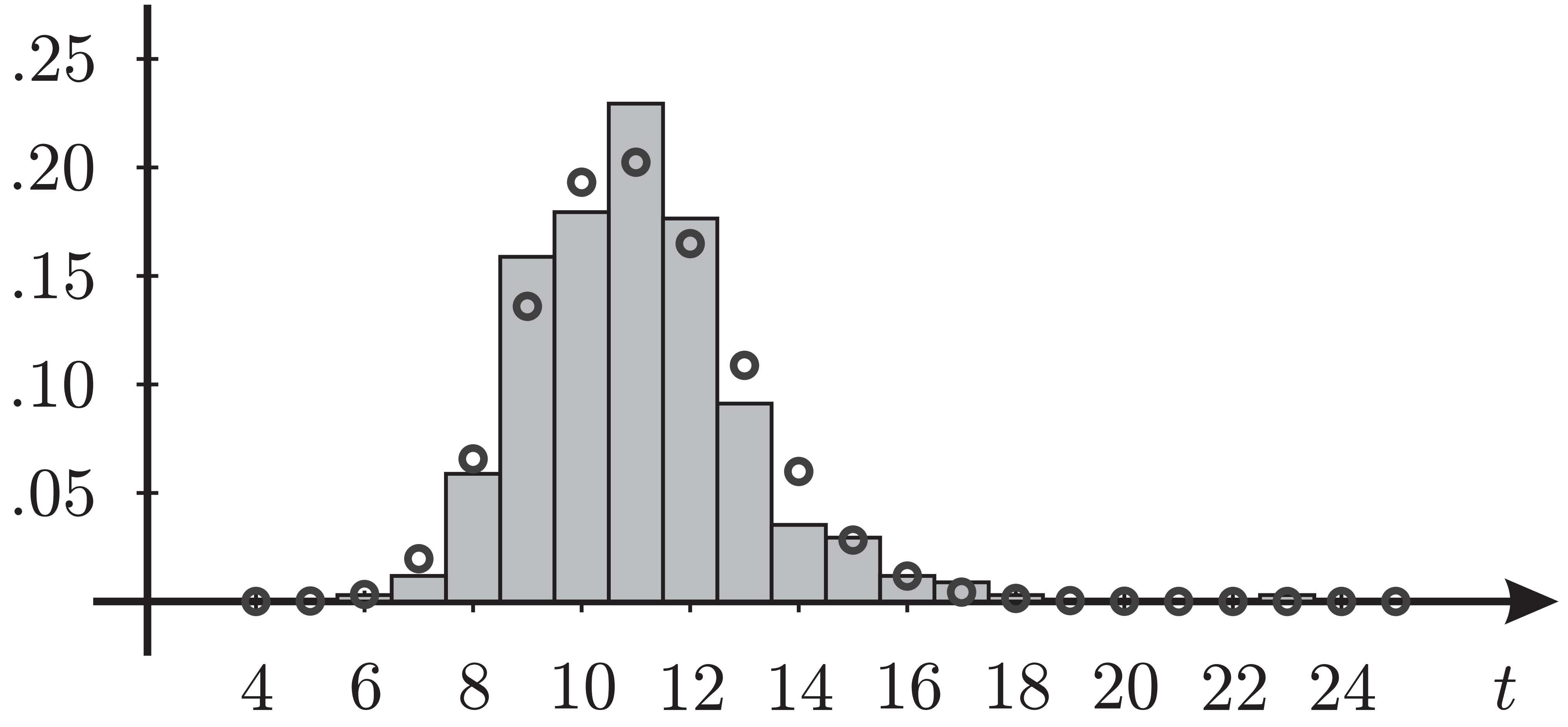


$$p(4) = 1/365,\ p(5) = 2/365,\ p(6) = 4/365,\ \ldots$$



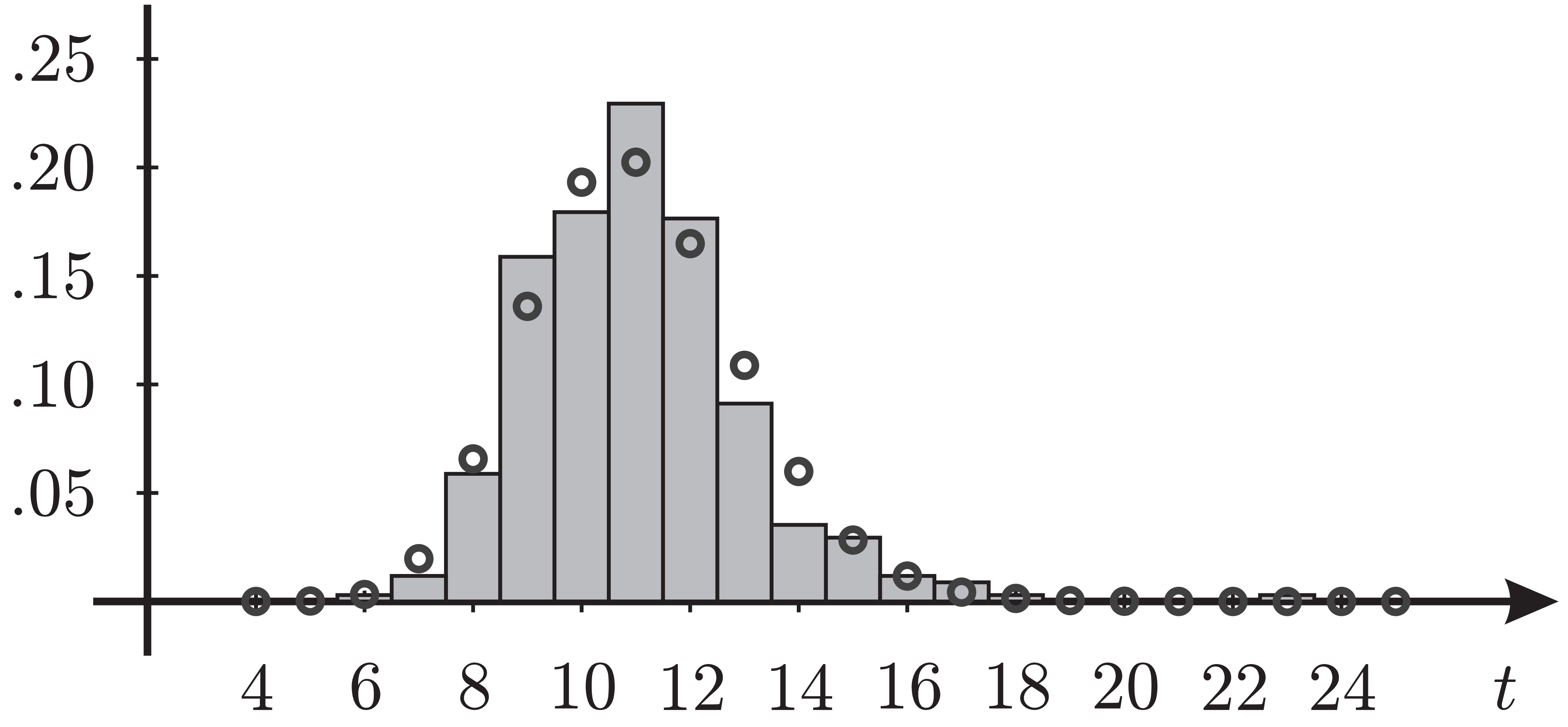
Continuous Commute Times

- It never actually takes *exactly* 12 minutes; I rounded each observation to the nearest integer number of minutes.
  - Actual data was 12.345 minutes, 11.78213 minutes, etc.



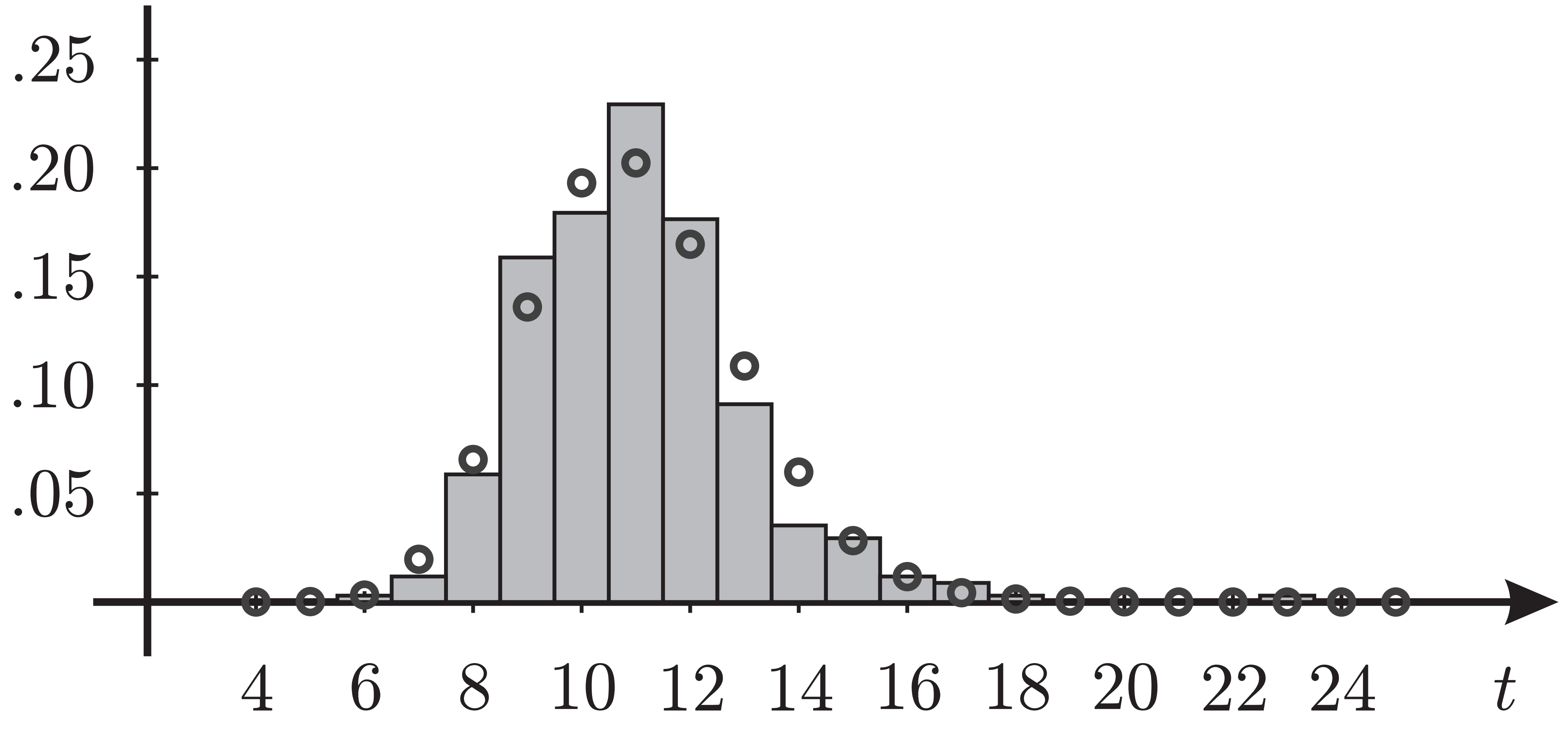
Continuous Commute Times

- It never actually takes *exactly* 12 minutes; I rounded each observation to the nearest integer number of minutes.
  - Actual data was 12.345 minutes, 11.78213 minutes, etc.
- **Question:** Could we use a Poisson distribution to predict the *exact* commute time (rather than the nearest number of minutes)? Why?



Using Histograms

Consider the continuous commuting example again, with observations 12.345 minutes, 11.78213 minutes, etc.



- **Question:** What is the random variable?
- **Question:** How could we turn our observations into a histogram?

Probability Density Functions (PDFs)

Definition: Given a continuous sample space Ω and event space $\mathscr{E} = B(\Omega)$, any function $p : \Omega \to [0,\infty)$ satisfying $\int_\Omega p(\omega) d\omega = 1$ is a **probability density function**.

- For a continuous sample space, instead of defining P directly, we can define a **probability density function** $p : \Omega \to [0,\infty)$.
- The probability for any event $A \in \mathscr{E}$ is then defined as

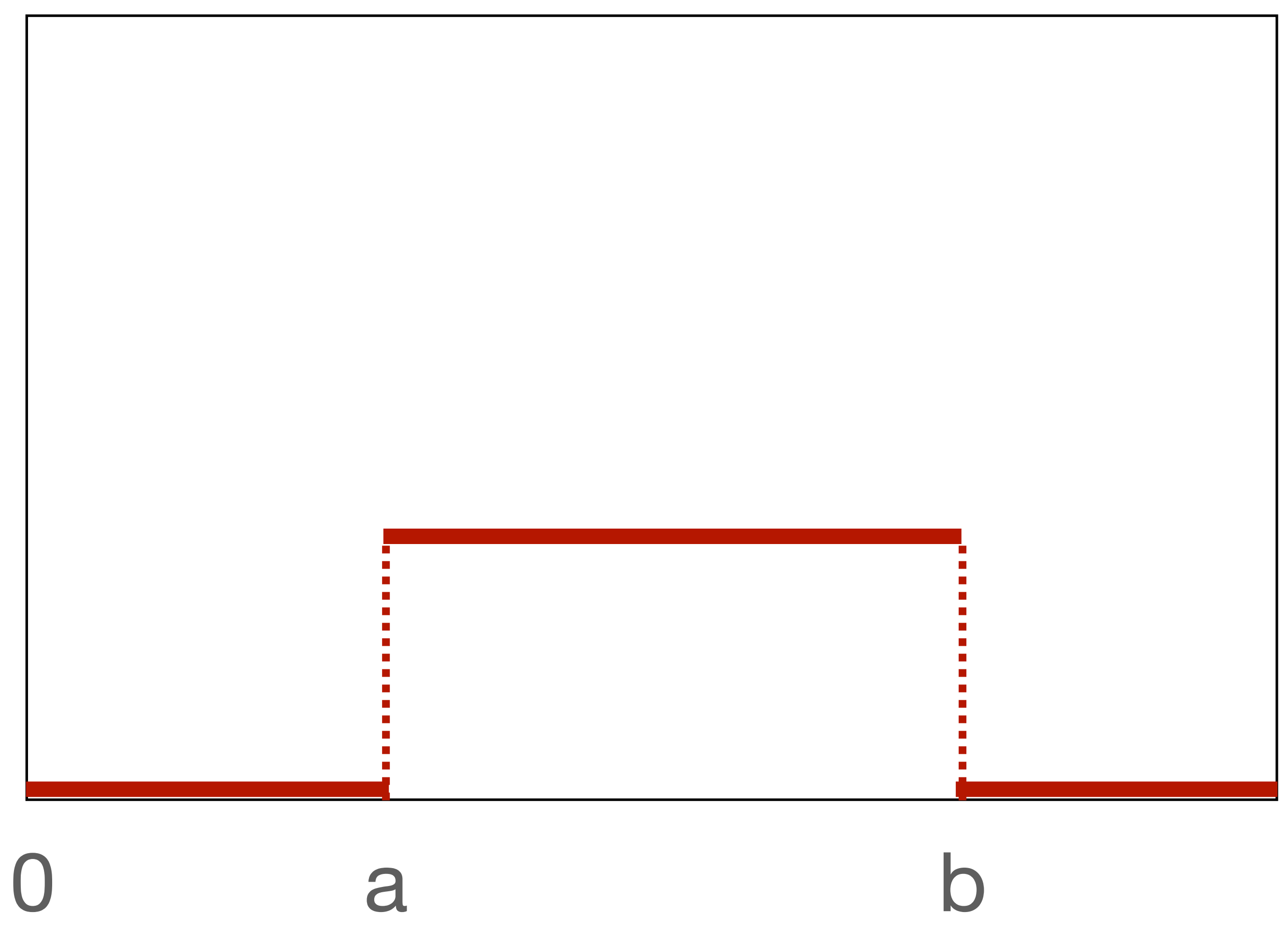
$$P(A) = \int_A p(\omega) d\omega.$$

Useful PDFs: Uniform

A **uniform distribution** is a distribution over a real interval. It has two parameters: a and b.

$$\Omega = [a, b]$$

$$p(\omega) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le \omega \le b, \\ 0 & \text{otherwise.} \end{cases}$$



Question: Does Ω have to be bounded?

Exercise: Check that the uniform pdf satisfies the required properties

Recall that the antiderivative of 1 is x, because the derivative of x is 1

$$\int_a^b p(x)dx = \int_a^b \frac{1}{b-a}dx$$

$$= \frac{1}{b-a}\int_a^b dx = \frac{1}{b-a}x\,|_a^b$$

$$= \frac{1}{b-a}(b-a) = 1$$

Useful PDFs: Gaussian

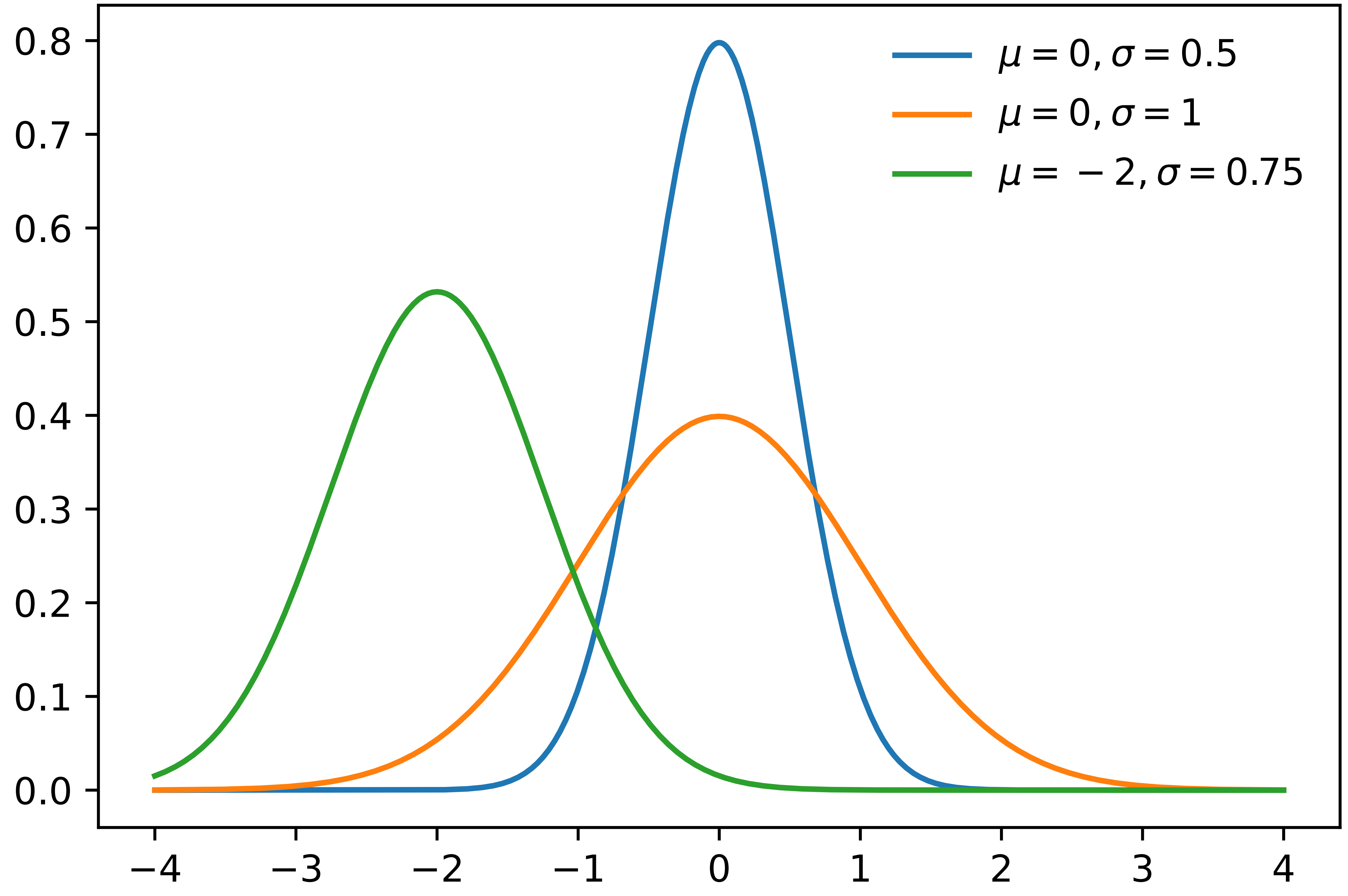
A **Gaussian distribution** is a distribution over the real numbers. It has two parameters: $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$.

$$\Omega = \mathbb{R}$$

$$p(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(\omega - \mu)^2\right)$$

where $\exp(x) = e^x$

Also called a normal distribution and written $\mathcal{N}(\mu, \sigma^2)$



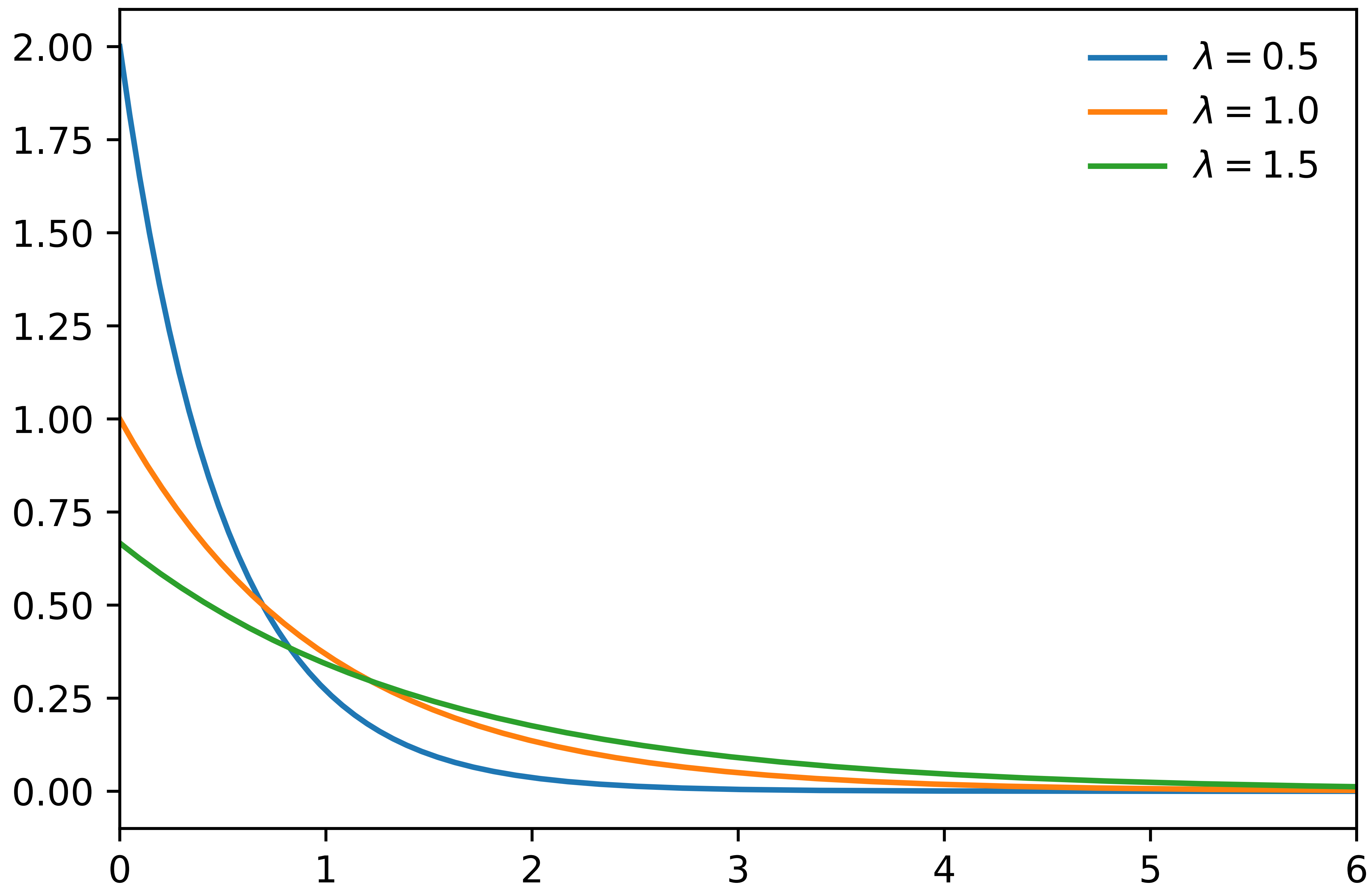
Useful PDFs: Exponential

An **exponential distribution** is a distribution over the positive reals. It has one parameter $\lambda > 0$.

$$\Omega = \mathbb{R}$$

$$p(\omega) = \lambda \exp(-\lambda\omega)$$

1 is here!



Why can the density be above 1?

Consider an interval event $A = [x, x + \Delta x]$, for small Δx.

$$P(A) = \int_x^{x+\Delta x} p(\omega)\, d\omega .$$
$$\approx p(x)\Delta x$$

- $p(x)$ can be big, because Δx can be very small
  - In particular, $p(x)$ can be bigger than 1
- But $P(A)$ **must** be less than or equal to 1

PMFs vs PDFs

1. When sample space Ω is **discrete:**
   - Singleton event: $P(\{\omega\}) = p(\omega)$ for $\omega \in \Omega$

$$P(A) = \sum_{\omega \in \Omega} p(\omega)$$

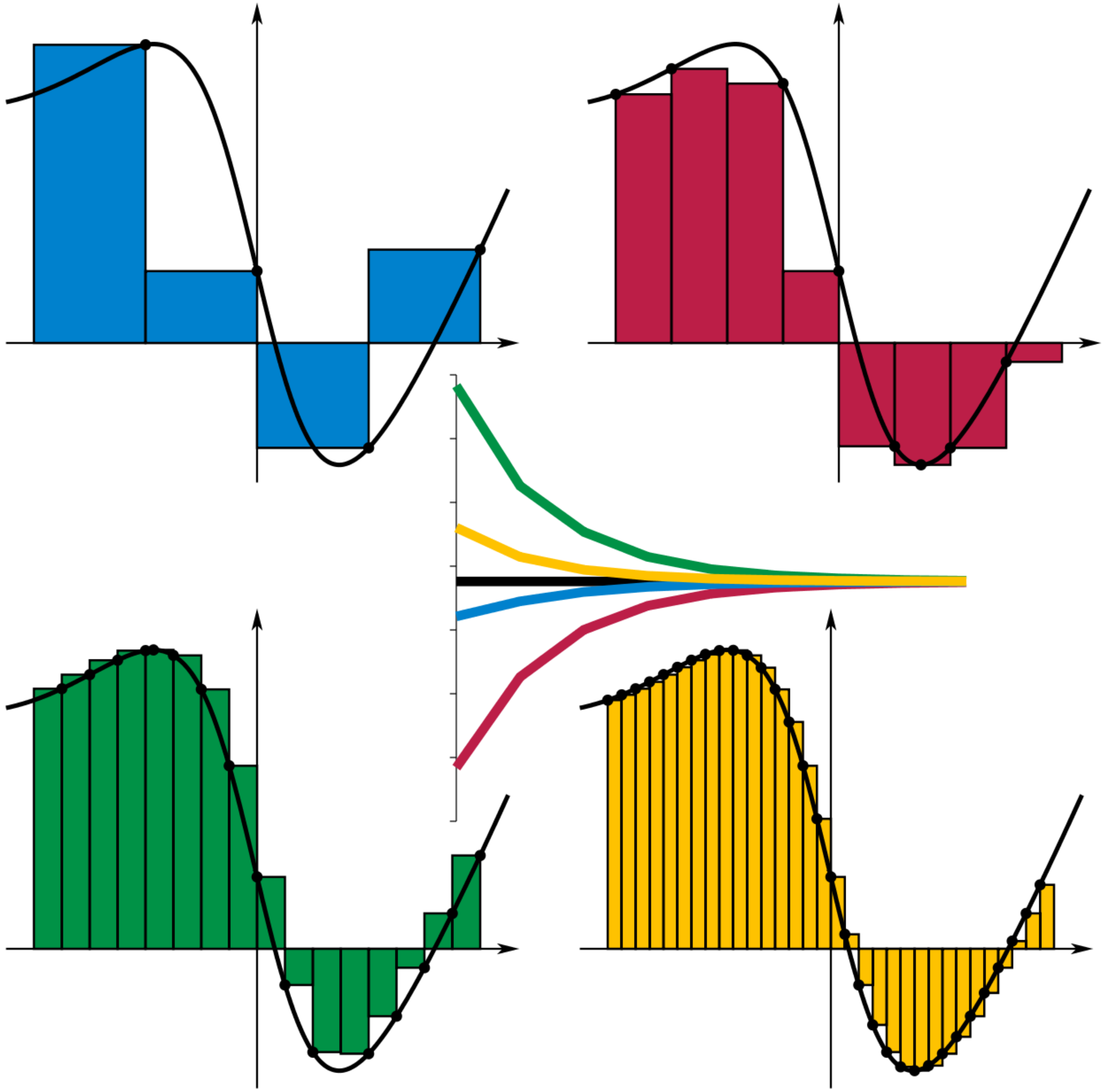
2. When sample space Ω is **continuous:**
   - Example: Stopping time for a car with $\Omega = [3,12]$
   - **Question:** What is the probability that the stopping time is *exactly* 3.14159?

$$P(A) = \int_A p(\omega) d\omega$$

$$P(\{3.14159\}) = \int_{3.14159}^{3.14159} p(\omega) d\omega$$

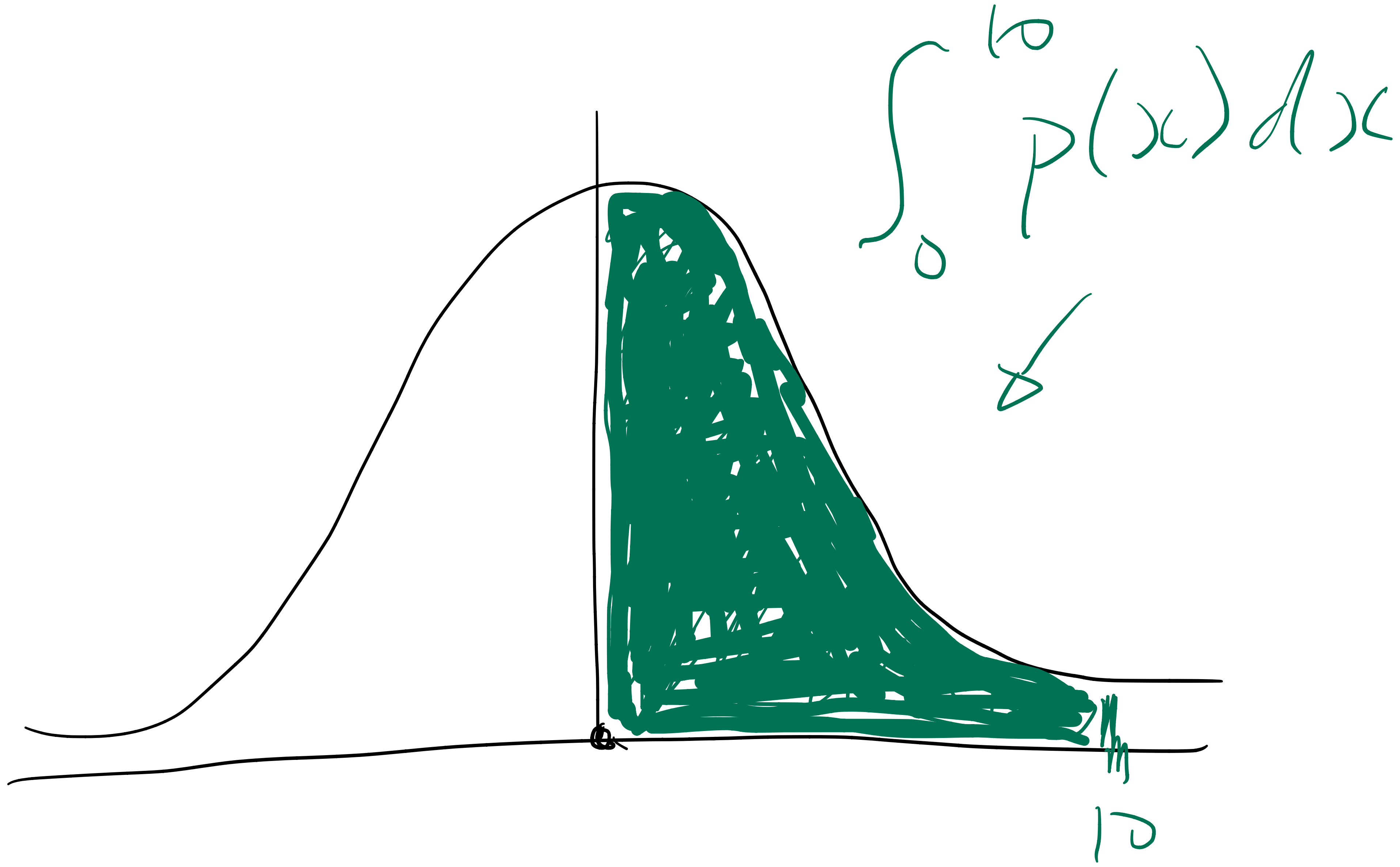
   - More reasonable: Probability that stopping time is between 3 to 3.5.

Recall Integration



Integration to give the probability of an event

- Imagine the PDF looks like the following concave function



$$p(0 \leq X \leq 10) = \int_0^{10} p(x)dx$$

Area under the curve reflects the probability of seeing an outcome in that region

Example comparing integration and summation

Exercise

- Imagine I asked you to tell me the probability that my birthday is on February 10 or July 9.
  - What is the outcome space and what is the event for this question?
  - Would we use a PMF or PDF to model these probabilities?
- Imagine I asked you to tell me the probability that the Uber would be here in between 3-5 minutes
  - What is the outcome space and what is the event for this question?
  - Would we use a PMF or PDF to model these probabilities?

Summary

- Probabilities are a means of **quantifying uncertainty**
- A probability distribution is defined on a measurable space consisting of a **sample space** and an **event space**.
- **Discrete** sample spaces (and random variables) are defined in terms of **probability mass functions** (PMFs)
- **Continuous** sample spaces (and random variables) are defined in terms of **probability density functions** (PDFs)
- **Random variables** are more convenient than operating directly on probability spaces