CMPUT 267 Basics of Machine Learning
Winter 2024

January 16 2024

Announcements

- Please read FAQ document on course webpage.
- Course information at https://nidhihegde.github.io/mlbasics
- Assignment due dates
- TA Office hours - updated
- Participation - Reading Exercises
  - on eClass;
  - open for a 48 hour period; one hour to complete
  - first one is a practise one - just a pdf, not as a quiz on eClass
  - First one that counts open ~~Monday, closes (due) Tuesday 11:59 pm~~ Tuesday 10am and closes Thursday 10am, as mentioned on eClass, and you have 60 minutes to complete it.

Outline

1. Recap
2. Random Variables
3. Multiple Random Variables
4. Independence
5. Expectations and Moments

Recap

- Probabilities are a means of **quantifying uncertainty**
- The **probability space** models an experiment, or a real world process.
- The sample space Ω : the set of all possible outcomes of the experiment.
- The event space $\mathscr{E}$: $\mathscr{E} \subseteq \mathscr{P}(\Omega)$, the space of potential results of the experiment.
- A probability distribution is defined on a measurable space consisting of a sample space and an event space. Any function $P : \mathscr{E} \rightarrow [0,1]$ that is a probability measure.
- A probability distribution is defined on a measurable space consisting of a **sample space** and an **event space**.
- **Discrete** sample spaces (and random variables) are defined in terms of **probability mass functions** (PMFs)
- **Continuous** sample spaces (and random variables) are defined in terms of **probability density functions** (PDFs)

Discrete vs. Continuous Sample Spaces

Discrete (countable) outcomes

$\Omega = \{1,2,3,4,5,6\}$

$\Omega = \{\text{person, woman, man, camera, TV}, \ldots\}$

$\Omega = \mathbb{N}$

$\mathscr{E} = \{\emptyset, \{1,2\}, \{3,4,5,6\}, \{1,2,3,4,5,6\}\}$

Typically**:** $\mathscr{E} = \mathscr{P}(\Omega)$

Question:
$\mathscr{E} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$?

Continuous (uncountable) outcomes

$\Omega = [0,1]$

$\Omega = \mathbb{R}$

$\Omega = \mathbb{R}^k$

$\mathscr{E} = \{\emptyset, [0,0.5], (0.5,1.0], [0,1]\}$

Typically: $\mathscr{E} = B(\Omega)$ ("Borel field")

Note: *not* $\mathscr{P}(\Omega)$

Random Variables

Rather than referring to the probability space, we refer to probabilities on quantities of interest.

Example: Suppose we observe both a die's number, and where it lands.

$$\Omega = \{(left,1), (right,1), (left,2), (right,2), \ldots, (right,6)\}$$

We might want to think about the probability that we get a large number, without thinking about where it landed.

We could ask about $P(X \geq 4)$, where X = the number that comes up.

Random variables are a way of reasoning about a complicated underlying probability space in a more straightforward way.

Random Variables, Formally

Given a probability space $(\Omega, \mathscr{E}, P)$, a **random variable** is a function $X : \Omega \to \Omega_X$ (where Ω_X is some other outcome space), satisfying

$$\{\omega \in \Omega \mid X(\omega) \in A\} \in \mathscr{E} \quad \forall A \in B(\Omega_X).$$

It follows that $P_X(A) = P(\{\omega \in \Omega \mid X(\omega) \in A\})$.

Example: Let Ω be a population of people, and $X(\omega)$ = height, and $A = [5'1'', 5'2'']$.

$$P(X \in A) = P(5'1'' \leq X \leq 5'2'') = P(\{\omega \in \Omega : X(\omega) \in A\}).$$

Random Variables and Events

- A Boolean expression involving random variables defines an event:

  E.g., $P(X \geq 4) = P(\{\omega \in \Omega \mid X(\omega) \geq 4\})$

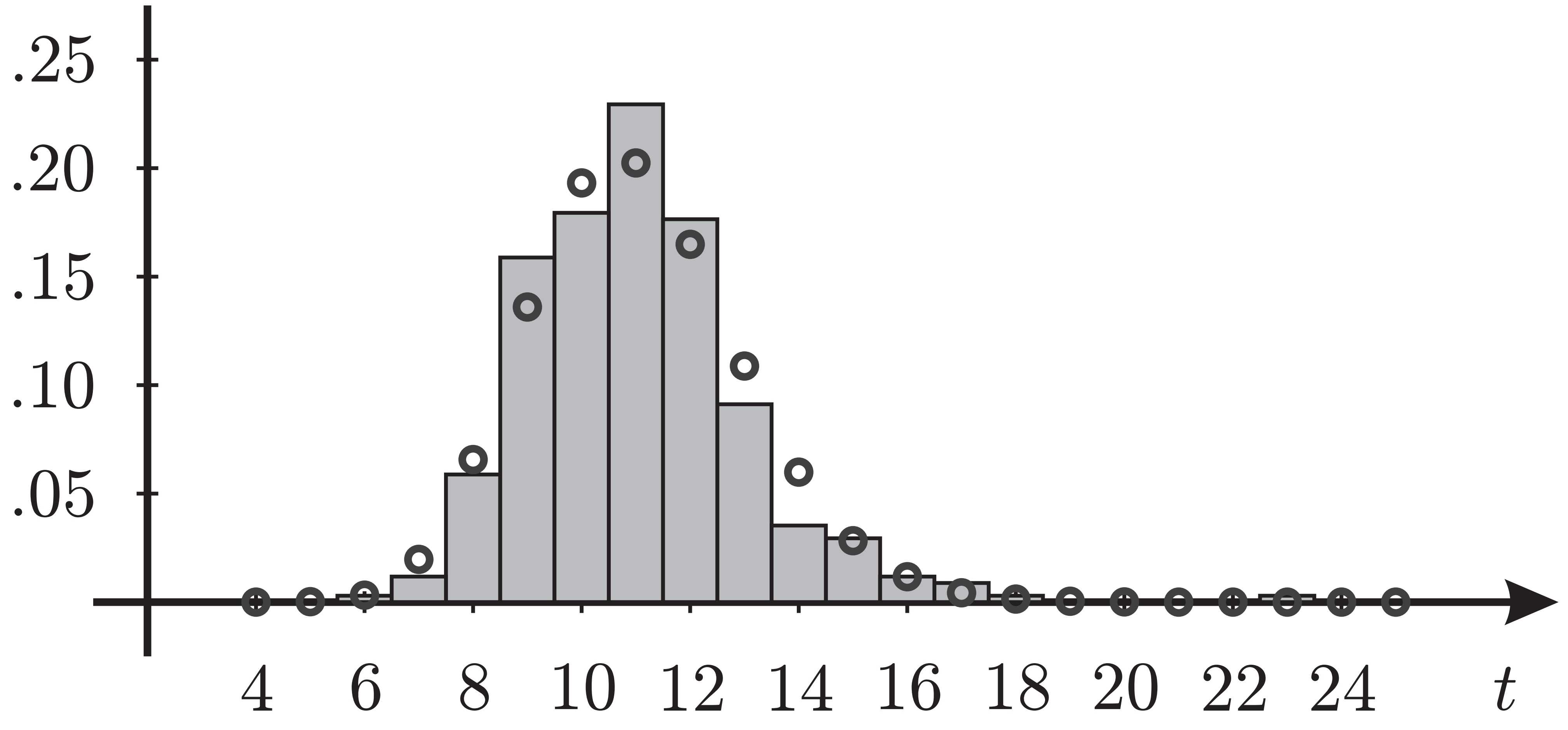
- Similarly, every event can be understood as a Boolean random variable:

$$Y = \begin{cases} 1 & \text{if event } A \text{ occurred} \\ 0 & \text{otherwise.} \end{cases}$$

- From this point onwards, we will exclusively reason in terms of random variables rather than probability spaces.

Example: Histograms

Consider the continuous commuting example again, with observations 12.345 minutes, 11.78213 minutes, etc.



- **Question:** What is the random variable?
- **Question:** How could we turn our observations into a histogram?

What About Multiple Variables?

- So far, we've really been thinking about a single random variable at a time
- Straightforward to define multiple random variables on a single probability space

Example: Suppose we observe both a die's number, and where it lands.

$$\Omega = \{(left, 1), (right, 1), (left, 2), (right, 2), \ldots, (right, 6)\}$$

$X(\omega) = \omega_2 =$ number

$$Y(\omega) = \left\{\begin{array}{ll} 1 & \text{if } \omega_1 = left \\ 0 & \text{otherwise.} \end{array}\right\} = 1 \text{ if landed on left}$$

$$P(Y = 1) = P(\{\omega \mid Y(\omega) = 1\})$$

$$P(X \geq 4 \land Y = 1) = P(\{\omega \mid X(\omega) \geq 4 \land Y(\omega) = 1\})$$

Joint Distribution

We typically model the **interactions** of different random variables.

Joint probability mass function: $p(x, y) = P(X = x, Y = y)$

$$\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) = 1$$

Example: $\mathcal{X} = \{0,1\}$ (young, old) and $\mathcal{Y} = \{0,1\}$ (no arthritis, arthritis)

	Y=0	**Y=1**
X=0	P(X=0,Y=0) = 1/2	P(X=0, Y=1) = 1/100
X=1	P(X=1, Y=0) = 1/10	P(X=1, Y=1) = 39/100

Is this joint distribution valid?

Example: $\mathscr{X} = \{0,1\}$ (young, old) and $\mathscr{Y} = \{0,1\}$ (no arthritis, arthritis)

	Y=0	**Y=1**
X=0	P(X=0,Y=0) = 50/100	P(X=0, Y=1) = 1/100
X=1	P(X=1, Y=0) = 10/100	P(X=1, Y=1) = 39/100

- **Exercise**: Check if $\sum_{x\in\{0,1\}} \sum_{y\in\{0,1\}} p(x, y) = 1$

$$\sum_{x\in\{0,1\}} \sum_{y\in\{0,1\}} p(x, y) = 1/2 + 1/100 + 1/10 + 39/100 = 1$$

Questions About Multiple Variables

Example: $\mathscr{X} = \{0,1\}$ (young, old) and $\mathscr{Y} = \{0,1\}$ (no arthritis, arthritis)

	Y=0	**Y=1**
X=0	P(X=0,Y=0) = 1/2	P(X=0, Y=1) = 1/100
X=1	P(X=1, Y=0) = 1/10	P(X=1, Y=1) = 39/100

- Are these two variables related at all? Or do they change **independently**?
- Given this distribution, can we determine the distribution over just Y? I.e., what is $P(Y=1)$? (**marginal distribution**)
- If we knew something about one variable, does that tell us something about the distribution over the other? E.g., if I know $X=0$ (person is young), does that tell me the prob. that person we know is young has arthritis? (**conditional probability** $P(Y=1 \mid X=1)$)

Marginal Distribution for Y

$$p(Y=0) = \sum_{x \in \mathcal{X}} p(x,0) = \sum_{x \in \{\text{young,old}\}} p(x,0)$$

$$p(Y=1) = \sum_{x \in \mathcal{X}} p(x,1) = \sum_{x \in \{\text{young,old}\}} p(x,1)$$

	Y=0	**Y=1**
X=0	P(X=0,Y=0) = 1/2	P(X=0, Y=1) = 1/100
X=1	P(X=1, Y=0) = 1/10	P(X=1, Y=1) = 39/100

More generically

$$p(y) = \sum_{x \in \mathcal{X}} p(x,y)$$

Back to our example

Example: $\mathscr{X} = \{0,1\}$ (young, old) and $\mathscr{Y} = \{0,1\}$ (no arthritis, arthritis)

	Y=0	**Y=1**
X=0	P(X=0,Y=0) = 50/100	P(X=0, Y=1) = 1/100
X=1	P(X=1, Y=0) = 10/100	P(X=1, Y=1) = 39/100

- **Exercise**: Compute marginal $p(x) = \sum_{y \in \{0,1\}} p(x, y)$

Back to our example (cont)

Example: $\mathscr{X} = \{0,1\}$ (young, old) and $\mathscr{Y} = \{0,1\}$ (no arthritis, arthritis)

	Y=0	**Y=1**
X=0	P(X=0,Y=0) = 50/100	P(X=0, Y=1) = 1/100
X=1	P(X=1, Y=0) = 10/100	P(X=1, Y=1) = 39/100

- **Exercise**: Compute marginal $p(x=1) = \sum_{y \in \{0,1\}} p(x=1, y) = 49/100,$

$p(x=0) = 1 - p(x=1) = 51/100$

Marginal distributions

- For two random variables X, Y,
- If they are discrete we have $p(x) = \sum_{y \in \mathcal{Y}} p(x, y)$
- If they are continuous we have $p(x) = \int_{\mathcal{Y}} p(x, y) dy$
- If X is discrete and Y is continuous then $p(x) = \int_{\mathcal{Y}} p(x, y) dy$
- If X is continuous and Y is discrete then $p(x) = \sum_{y \in \mathcal{Y}} p(x, y)$

Marginal Distributions

A **marginal distribution** is defined for a subset of $\overrightarrow{X}$ by summing or integrating out the remaining variables. (We will often say that we are "marginalizing over" or "marginalizing out" the remaining variables).

Discrete case: $p(x_i) = \sum_{x_1 \in \mathcal{X}_1} \cdots \sum_{x_{i-1} \in \mathcal{X}_{i-1}} \sum_{x_{i+1} \in \mathcal{X}_{i+1}} \cdots \sum_{x_d \in \mathcal{X}_d} p(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$

Continuous: $p(x_i) = \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_{i-1}} \int_{\mathcal{X}_{i+1}} \cdots \int_{\mathcal{X}_d} p(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)\, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$

Question: Why do we write p for $p(x_i)$ and $p(x_1, \ldots, x_d)$?

- They can't be the same function, they have different domains!

Are these really the same function?

- **No.** They're not the same function.
- But they are **derived** from the **same joint distribution**.
- So for brevity we will write $p(x, y)$, $p(x)$ and $p(y)$
- Even though it would be more precise to write something like $p(x, y)$, $p_x(x)$ and $p_y(y)$
- We can tell which function we're talking about from context (i.e., arguments)

PMFs and PDFs of Many Variables

In general, we can consider a d-dimensional random variable $\vec{X} = (X_1, \ldots, X_d)$ with vector-valued outcomes $\vec{x} = (x_1, \ldots, x_d)$, with each x_i chosen from some $\mathcal{X}_i$. Then,

Discrete case:

$p : \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_d \to [0,1]$ is a **(joint) probability mass function** if

$$\sum_{x_1 \in \mathcal{X}_1} \sum_{x_2 \in \mathcal{X}_2} \cdots \sum_{x_d \in \mathcal{X}_d} p(x_1, x_2, \ldots, x_d) = 1$$

Continuous case:

$p : \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_d \to [0,\infty)$ is a **(joint) probability density function** if

$$\int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \int_{\mathcal{X}_d} p(x_1, x_2, \ldots, x_d)\, dx_1 dx_2 \ldots dx_d = 1$$

Rules of Probability Already Covered the Multidimensional Case

Outcome space is $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_d$

Outcomes are multidimensional variables $\mathbf{x} = [x_1, x_2, \ldots, x_d]$

Discrete case:

$p : \mathcal{X} \to [0,1]$ is a **(joint) probability mass function** if $\sum_{\mathbf{x} \in \mathcal{X}} p(\mathbf{x}) = 1$

Continuous case:

$p : \mathcal{X} \to [0,\infty)$ is a **(joint) probability density function** if $\int_{\mathcal{X}} p(\mathbf{x})\, d\mathbf{x} = 1$

But useful to recognize that we have multiple variables

Conditional Distribution

Definition: **Conditional probability distribution**

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)}$$

This same equation will hold for the corresponding PDF or PMF:

$$p(y \mid x) = \frac{p(x, y)}{p(x)}$$

Question: if $p(x, y)$ is small, does that imply that $p(y \mid x)$ is small?

Visualizing the conditional distribution

$P(X = \text{young} \mid Y = 0) = P(X = \text{young}, Y = 0)/P(Y = 0) = (50/100)/(60/100) = 50/60$

Chain Rule

From the definition of conditional probability:

$$
\begin{aligned}
p(y \mid x) &= \frac{p(x, y)}{p(x)} \\
\iff p(y \mid x)p(x) &= \frac{p(x, y)}{p(x)}p(x) \\
\iff p(y \mid x)p(x) &= p(x, y)
\end{aligned}
$$

This is called the **Chain Rule**.

Multiple Variable Chain Rule

The chain rule generalizes to multiple variables:

$$p(x, y, z) = p(x, y \mid z)p(z) = p(x \mid y, z)\underbrace{p(y \mid z)p(z)}_{p(y,z)}$$

Definition: **Chain rule**

$$p(x_1, \ldots, x_d) = p(x_d)\prod_{i=1}^{d-1} p(x_i \mid x_{i+1}, \ldots x_d)$$

$$= p(x_1)\prod_{i=2}^{d} p(x_i \mid x_1, \ldots x_{i-1})$$

The Order Does Not Matter

The RVs are not ordered, so we can write

$$
\begin{aligned}
p(x, y, z) &= p(x \mid y, z)p(y|z)p(z) \\
&= p(x \mid y, z)p(z|y)p(y) \\
&= p(y \mid x, z)p(x|z)p(z) \\
&= p(y \mid x, z)p(z|x)p(x) \\
&= p(z \mid x, y)p(y|x)p(x) \\
&= p(z \mid x, y)p(x|y)p(y)
\end{aligned}
$$

All of these probabilities are equal

Bayes' Rule

From the chain rule, we have:

$$\begin{aligned} p(x, y) &= p(y \mid x)p(x) \\ &= p(x \mid y)p(y) \end{aligned}$$

- Often, $p(x \mid y)$ is easier to compute than $p(y \mid x)$
  - e.g., where x is **features** and y is **label**

Definition: Bayes' rule

$$p(y \mid x) = \frac{p(x \mid y)p(y)}{p(x)}$$

Bayes' Rule

- Bayes’ rule is typically used to reason about our beliefs, given new information
- Example: a scientist might have a belief about the prevalence of cancer in smokers (Y), and update with new evidence (X)
- In ML: we have a belief over our estimator (Y), and we update with new data that is like new evidence (X)

Definition: Bayes' rule

$$p(y \mid x) = \frac{p(x \mid y)p(y)}{p(x)}$$

Posterior: $p(y \mid x)$

Likelihood: $p(x \mid y)$

Prior: $p(y)$

Evidence: $p(x)$

Example:
Drug Test

$$p(y \mid x) = \frac{p(x \mid y)p(y)}{p(x)}$$

Posterior: $p(y \mid x)$
Likelihood: $p(x \mid y)$
Prior: $p(y)$
Evidence: $p(x)$

Example:

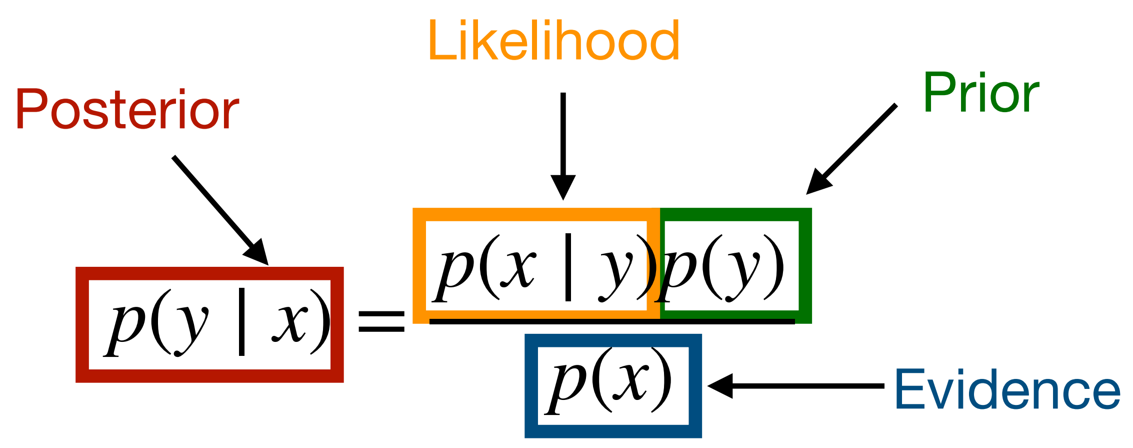
$$p(Test = pos \mid Drug = T) = 0.99$$
$$p(Test = pos \mid Drug = F) = 0.01$$
$$p(Drug = True) = 0.005$$

Questions:

1. What is $p(Drug = F)$?
2. What is $p(Drug = T \mid Test = pos)$?

Mapping to the formula, let
X be Test
Y be presence of the drug

Example:
Drug Test



Example:

$$p(Test = pos \mid Drug = T) = 0.99$$

$$p(Test = pos \mid Drug = F) = 0.01$$

$$p(Drug = True) = 0.005$$

Questions:

1. What is $p(Drug = F)$?
2. What is $p(Drug = T \mid Test = pos)$?

$$p(Drug = F) = 1 - p(Drug = T) = 1 - 0.005 = 0.995$$

Example: Drug Test

$$p(y \mid x) = \frac{p(x \mid y)p(y)}{p(x)}$$

Posterior: $p(y \mid x)$; Likelihood: $p(x \mid y)$; Prior: $p(y)$; Evidence: $p(x)$

Example:

$$p(Test = pos \mid Drug = T) = 0.99$$
$$p(Test = pos \mid Drug = F) = 0.01$$
$$p(Drug = True) = 0.005$$

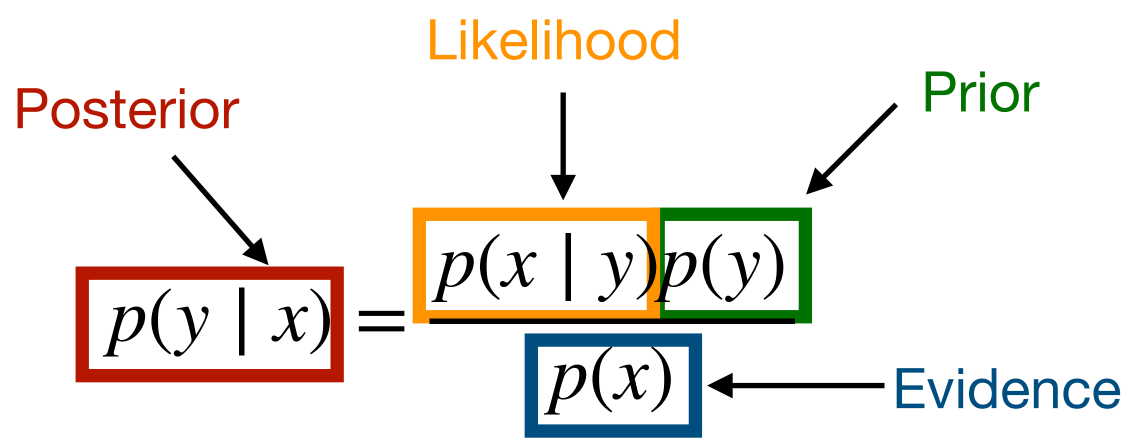
Questions:

1. What is $p(Drug = F)$?
2. What is $p(Drug = T \mid Test = pos)$?

$$p(Drug = T \mid Test = pos) = \frac{p(Test = pos \mid Drug = T)p(Drug = T)}{p(Test = pos)}$$

Need to compute this part ($p(Test = pos)$)

Example: Drug Test



Example:

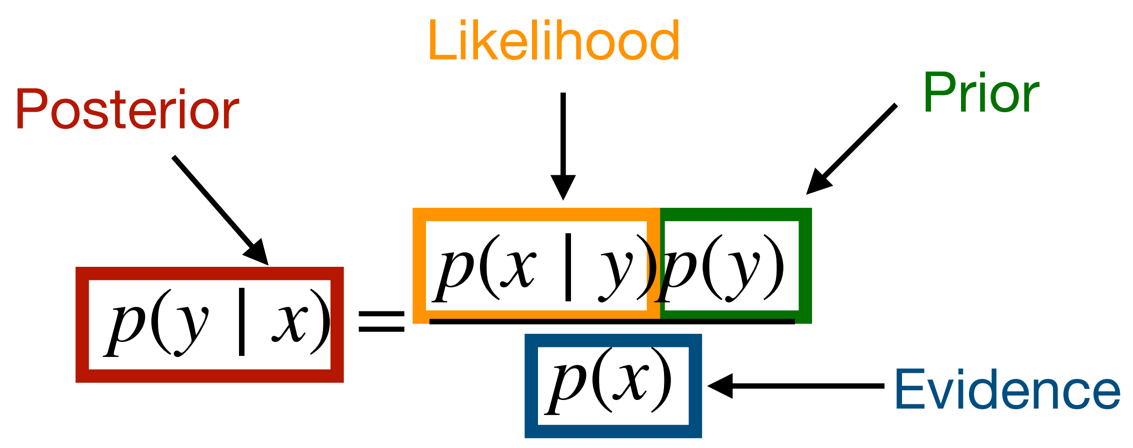
$$p(Test = pos \mid Drug = T) = 0.99$$
$$p(Test = pos \mid Drug = F) = 0.01$$
$$p(Drug = True) = 0.005$$

Questions:

1. What is $p(Drug = F)$?
2. What is $p(Drug = T \mid Test = pos)$?

$$p(Test = pos) = \sum_{d \in \{T,F\}} p(Test = pos, d)$$
$$= p(Test = pos, D = F) + p(Test = pos, D = T)$$
$$= p(Test = pos \mid D = F)p(D = F) + p(Test = pos \mid D = T)p(D = T)$$
$$= 0.03 \times 0.995 + 0.99 \times 0.005 = 0.0348$$

Example: Drug Test



Example:

$$p(Test = pos \mid Drug = T) = 0.99$$
$$p(Test = pos \mid Drug = F) = 0.01$$
$$p(Drug = True) = 0.005$$

Questions:

1. What is $p(Drug = F)$?
2. What is $p(Drug = T \mid Test = pos)$?

$$p(Test = pos) = 0.0348$$

$$p(Drug = T \mid Test = pos) = \frac{p(Test = pos \mid Drug = T)p(Drug = T)}{p(Test = pos)} = \frac{0.99 \times 0.005}{0.0348} \approx 0.142$$

Independence of Random Variables

Definition: X and Y are **independent** if:

$$p(x, y) = p(x)p(y)$$

X and Y are **conditionally independent given Z** if:

$$p(x, y \mid z) = p(x \mid z)p(y \mid z)$$

Example: Coins
(Ex. 9 in the course text)

- Suppose you have a biased coin: the probability that it comes up heads is not 0.5. Instead, it has some probability to *more* likely to come up heads.
- Let Z be the bias of the coin, with $\mathscr{Z} = \{0.3, 0.5, 0.8\}$ and probabilities $P(Z = 0.3) = 0.7$, $P(Z = 0.5) = 0.2$ and $P(Z = 0.8) = 0.1$.
  - **Question:** What other outcome space could we consider?
  - **Question:** What kind of distribution is this?
  - **Question:** What other kinds of distribution could we consider?
- Let X and Y be two consecutive flips of the coin
- **Question:** Are X and Y independent?
- **Question:** Are X and Y conditionally independent given Z?

Example: Coins (2)

- Now imagine I told you $Z = 0.3$ (i.e., probability of heads is 0.3)
- Let X and Y be two consecutive flips of the coin
- What is $P(X = Heads \mid Z = 0.3)$? What about $P(X = Tails \mid Z = 0.3)$?
- What is $P(Y = Heads \mid Z = 0.3)$? What about $P(Y = Tails \mid Z = 0.3)$?
- Is $P(X = x, Y = y \mid Z = 0.3) = P(X = x \mid Z = 0.3)P(Y = y \mid Z = 0.3)$?
    - That is, are X and Y conditionally independent given Z?

Example: Coins (3)

- Now imagine we do not know Z
  - e.g., you randomly grabbed it from a bin of coins with probabilities $P(Z = 0.3) = 0.7$, $P(Z = 0.5) = 0.2$ and $P(Z = 0.8) = 0.1$
- What is $P(X = Heads)$?

$$P(X = Heads) = \sum_{z \in \{0.3, 0.5, 0.8\}} P(X = Heads \,|\, Z = z) p(Z = z)$$

$$= P(X = Heads \,|\, Z = 0.3) p(Z = 0.3)$$

$$+ P(X = Heads \,|\, Z = 0.5) p(Z = 0.5)$$

$$+ P(X = Heads \,|\, Z = 0.8) p(Z = 0.8)$$

$$= 0.3 \times 0.7 + 0.5 \times 0.2 + 0.8 \times 0.1 = 0.39$$

Example: Coins (4)

- Now imagine we do not know Z
    - e.g., you randomly grabbed it from a bin of coins with probabilities $P(Z = 0.3) = 0.7$, $P(Z = 0.5) = 0.2$ and $P(Z = 0.8) = 0.1$
- Is $P(X = Heads, Y = Heads) = P(X = Heads)p(Y = Heads)$?
    - For brevity, lets use h for Heads

$$P(X = h, Y = h) = \sum_{z \in \{0.3, 0.5, 0.8\}} P(X = h, Y = h \,|\, Z = z)p(Z = z)$$

$$= \sum_{z \in \{0.3, 0.5, 0.8\}} P(X = h \,|\, Z = z)P(Y = h \,|\, Z = z)p(Z = z)$$

Example: Coins (4)

- $P(Z=0.3)=0.7$, $P(Z=0.5)=0.2$ and $P(Z=0.8)=0.1$
- Is $P(X=Heads, Y=Heads)=P(X=Heads)p(Y=Heads)$?

$$P(X=h, Y=h) = \sum_{z\in\{0.3,0.5,0.8\}} P(X=h, Y=h\,|\,Z=z)p(Z=z)$$

$$= \sum_{z\in\{0.3,0.5,0.8\}} P(X=h\,|\,Z=z)P(Y=h\,|\,Z=z)p(Z=z)$$

$$= P(X=h\,|\,Z=0.3)P(Y=h\,|\,Z=0.3)p(Z=0.3)$$

$$+P(X=h\,|\,Z=0.5)P(Y=h\,|\,Z=0.5)p(Z=0.5)$$

$$+P(X=h\,|\,Z=0.8)p(Y=h\,|\,Z=0.8)p(Z=0.8)$$

$$= 0.3\times 0.3\times 0.7 + 0.5\times \times 0.5\times 0.2 + 0.8\times 0.8\times 0.1$$

$$= 0.177 \neq 0.39 * 0.39 = 0.1521$$

Example: Coins (4)

- Let Z be the bias of the coin, with $\mathcal{Z} = \{0.3, 0.5, 0.8\}$ and probabilities $P(Z = 0.3) = 0.7$, $P(Z = 0.5) = 0.2$ and $P(Z = 0.8) = 0.1$.
- Let X and Y be two consecutive flips of the coin
- **Question:** Are X and Y conditionally independent given Z?
  - i.e., $P(X = x, Y = y \mid Z = z) = P(X = x \mid Z = z)P(Y = y \mid Z = z)$
- **Question:** Are X and Y independent?
  - i.e. $P(X = x, Y = y) = P(X = x)P(Y = y)$

The Distribution Changes Based on What We Know

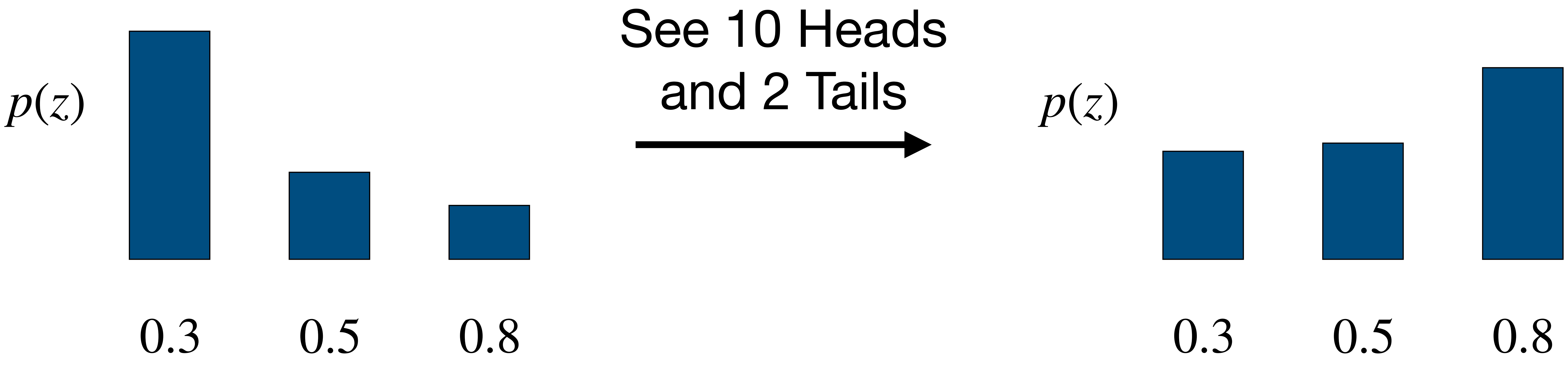
- The coin has some true bias z
- If we **know** that bias, we reason about $P(X = x \mid Z = z)$
  - Namely, the probability of x **given** we know the bias is z
- If we **do not know** that bias, then **from our perspective** the coin outcomes follows probabilities $P(X = x)$
  - The world still flips the coin with bias z
- Conditional independence is a property of the distribution we are reasoning about, not an objective truth about outcomes

A bit more intuition

- If we **do not know** that bias, then **from our perspective** the coin outcomes follows probabilities $P(X = x, Y = y)$
  - and X and Y are correlated
- If we know $X = h$, do we think it's more likely $Y = h$? i.e., is $P(X = h, Y = h) > P(X = h, Y = t)$?

Why is independence and conditional independence important?

- i.e., how is this relevant
- Let's imagine you want to infer (or learn) the bias of the coin, from data
  - data in this case corresponds to a sequence of flips $X_1, X_2, \ldots, X_n$
- You can ask: $P(Z = z \mid X_1 = H, X_2 = H, X_3 = T, \ldots, X_n = H)$



More uses for independence and conditional independence

- If I told you X = roof type was **independent** of Y = house price, would you use X as a feature to predict Y?
- Imagine you want to predict Y = Has Lung Cancer and you have an indirect correlation with X = Location since in Location 1 more people smoke on average. If you could measure Z = Smokes, then X and Y would be **conditionally independent** given Z.
  - Suggests you could look for such causal variables, that explain these correlations
- We will see the utility of conditional independence for learning models

Expected Value

The expected value of a random variable is the **weighted average** of that variable over its domain.

Definition: Expected value of a random variable

$$\mathbb{E}[X] = \begin{cases} \sum_{x \in \mathcal{X}} x p(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} x p(x)\, dx & \text{if } X \text{ is continuous.} \end{cases}$$

Relationship to Population Average and Sample Average

- Or Population Mean and Sample Mean
- Population Mean = Expected Value, Sample Mean estimates this number
  - e.g., Population Mean = average height of the entire population
- For RV X = height, p(x) gives the probability that a randomly selected person has height x
- Sample average: you randomly sample n heights from the population
  - implicitly you are sampling heights proportionally to p
- As n gets bigger, the sample average approaches the true expected value

Connection to Sample Average

- Imagine we have a biased coin, p(x = 1) = 0.75, p(x = 0) = 0.25
- Imagine we flip this coin 1000 times, and see (x = 1) 700 times
- The sample average is

$$\frac{1}{1000}\sum_{i=1}^{1000} x_i = \frac{1}{1000}\left[\sum_{i:x_i=0} x_i + \sum_{i:x_i=1} x_i\right] = 0 \times \frac{300}{1000} + 1 \times \frac{700}{1000} = = 0 \times 0.3 + 1 \times 0.7 = 0.7$$

- The true expected value is

$$\sum_{x \in \{0,1\}} p(x)x = 0 \times p(x=0) + 1p(x=1) = 0 \times 0.25 + 1 \times 0.75 = 0.75$$

Expected Value with Functions

The expected value of a function $f : \mathcal{X} \to \mathbb{R}$ of a random variable is the **weighted average** of that function's value over the domain of the variable.

Definition: Expected value of a function of a random variable

$$\mathbb{E}[f(X)] = \begin{cases} \sum_{x \in \mathcal{X}} f(x)p(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} f(x)p(x)\,dx & \text{if } X \text{ is continuous.} \end{cases}$$

Example:
Suppose you get $10 if heads is flipped, or lose $3 if tails is flipped.
What are your winnings **on expectation**?

Expected Value Example

Example:
Suppose you get \$10 if heads is flipped, or lose \$3 if tails is flipped.
What are your winnings **on expectation**?

X is the outcome of the coin flip, 1 for heads and 0 for tails

$$f(x) = \begin{cases} 3 & \text{if } x = 0 \\ 10 & \text{if } x = 1 \end{cases}$$

$Y = f(X)$ is a new random variable

$$\mathbb{E}[Y] = \mathbb{E}[f(X)] = \sum_{x \in \mathcal{X}} f(x)p(x) = f(0)p(0) + f(1)p(1) = .5 \times 3 + .5 \times 10 = 6.5$$

One More Example

Suppose X is the outcome of a dice role

$$f(x) = \begin{cases} -1 & \text{if } x \leq 3 \\ 1 & \text{if } x \geq 4 \end{cases}$$

$Y = f(X)$ is a new random variable. We see $Y = -1$ each time we observe 1, 2 or 3. We see $Y = 1$ each time we observe 4, 5, or 6.

$$\mathbb{E}[Y] = \mathbb{E}[f(X)] = \sum_{x \in \mathcal{X}} f(x)p(x)$$

$$= (-1)\Big(p(X=1) + p(X=2) + p(X=3)\Big)$$

$$+ \ (1)\Big(p(X=4) + p(X=5) + p(X=6)\Big)$$

One More Example

Suppose X is the outcome of a dice role

$$f(x) = \begin{cases} -1 & \text{if } x \leq 3 \\ 1 & \text{if } x \geq 4 \end{cases}$$

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We see $Y = 1$ each time we observe 4, 5, or 6.

$$\mathbb{E}[Y] = \mathbb{E}[f(X)] = \sum_{x \in \mathcal{X}} f(x)p(x) = \sum_{y \in \{-1,1\}} yp(y)$$

$$p(Y = -1) = p(X = 1) + p(X = 2) + p(X = 3) = 0.5$$

$$p(Y = 1) = p(X = 4) + p(X = 5) + p(X = 6) = 0.5$$

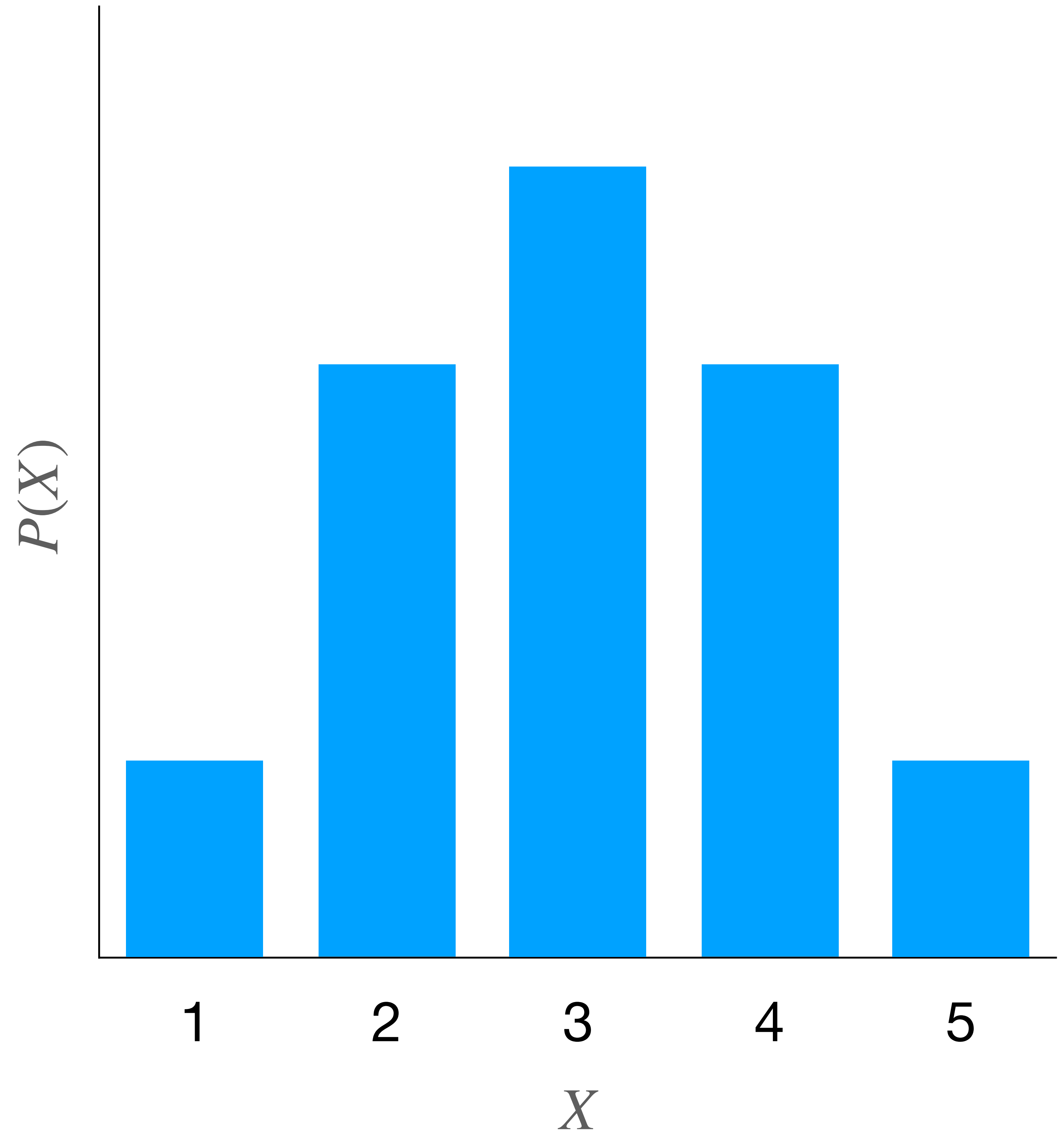
$$= (-1)\Big(p(X = 1) + p(X = 2) + p(X = 3)\Big)$$

$$+ \ (1)\Big(p(X = 4) + p(X = 5) + p(X = 6)\Big)$$

$$= -1(0.5) + 1(0.5)$$

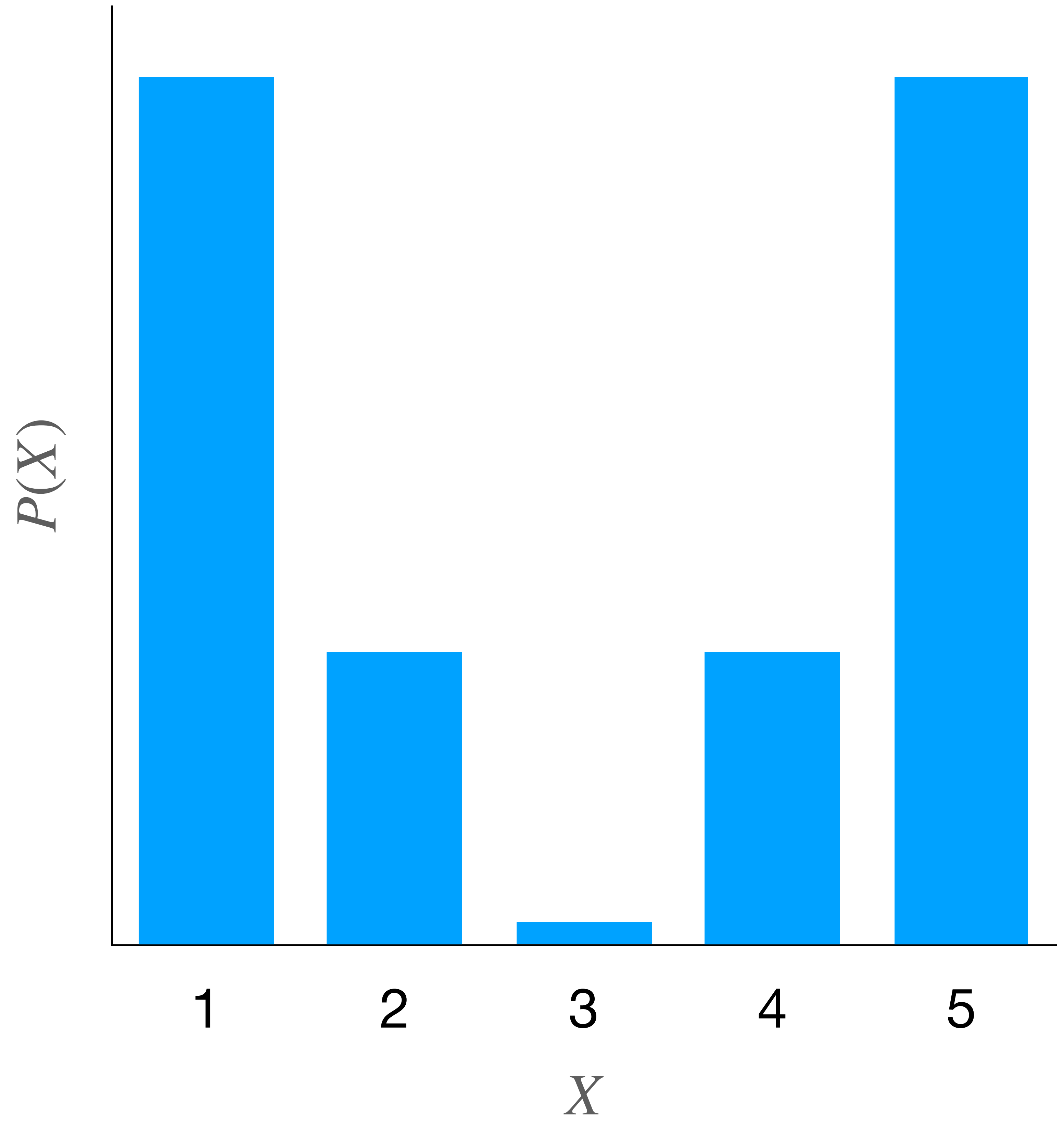
Summing over x with p(x) is equivalent, and simpler (no need to infer p(y))

Expected Value is a Lossy Summary



$$\mathbb{E}[X] = 3$$

$$\mathbb{E}[X^2] \simeq 10$$



$$\mathbb{E}[X] = 3$$

$$\mathbb{E}[X^2] \simeq 12$$

Conditional Expectations

Definition:

The **expected value of Y conditional on $X = x$** is

$$\mathbb{E}[Y \mid X = x] = \begin{cases} \sum_{y \in \mathcal{Y}} y p(y \mid x) & \text{if } Y \text{ is discrete,} \\ \int_{\mathcal{Y}} y p(y \mid x)\, dy & \text{if } Y \text{ is continuous.} \end{cases}$$

Question: What is $\mathbb{E}[Y \mid X]$?

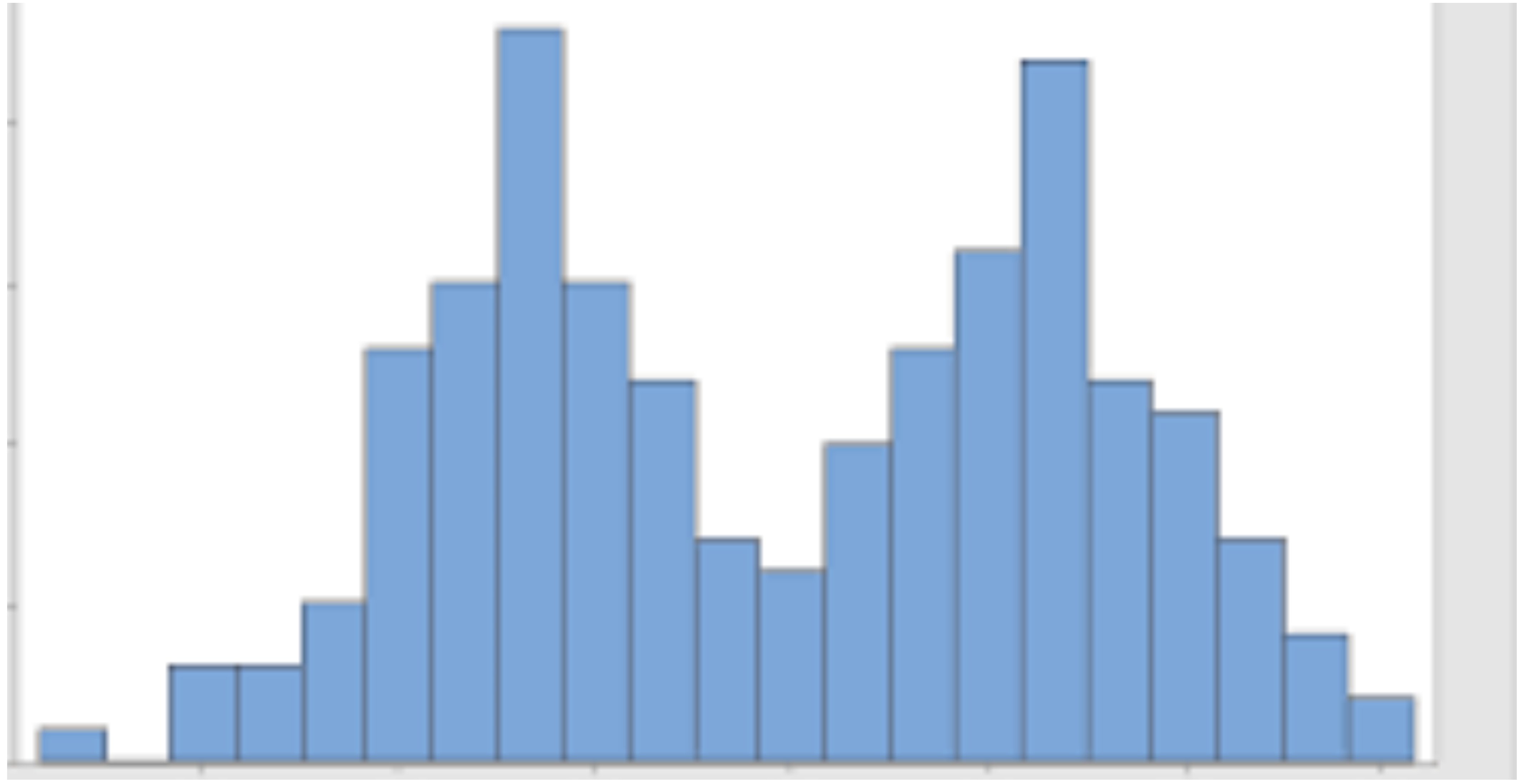
Conditional Expectation Example

- X is the type of a book, 0 for fiction and 1 for non-fiction
  - $p(X = 1)$ is the proportion of all books that are non-fiction
- Y is the number of pages
  - $p(Y = 100)$ is the proportion of all books with 100 pages
- $\mathbb{E}[Y|X = 0]$ is different from $\mathbb{E}[Y|X = 1]$
  - e.g. $\mathbb{E}[Y|X = 0] = 70$ is different from $\mathbb{E}[Y|X = 1] = 150$
- Another example: $\mathbb{E}[X|Z = 0.3]$ the expected outcome of the coin flip given that the bias is 0.3 ($\mathbb{E}[X|Z = 0.3] = 0 \times 0.7 + 1 \times 0.3 = 0.3$)

Conditional Expectation Example (cont)

- What do we mean by $p(y \mid X = 0)$? How might it differ from $p(y \mid X = 1)$

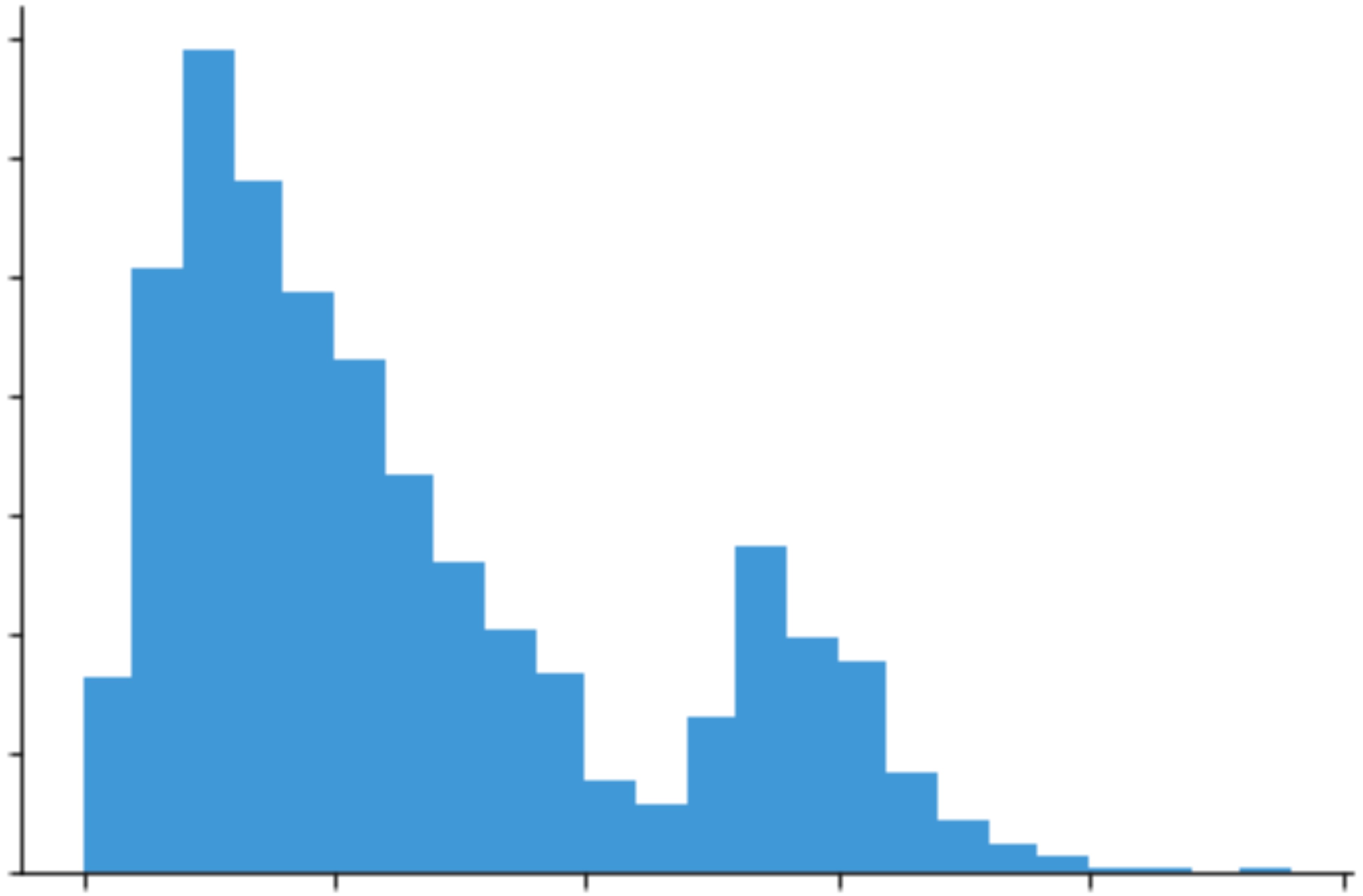
p(y) for X = 0, fiction books



Lots of shorter books

Lots of medium length books

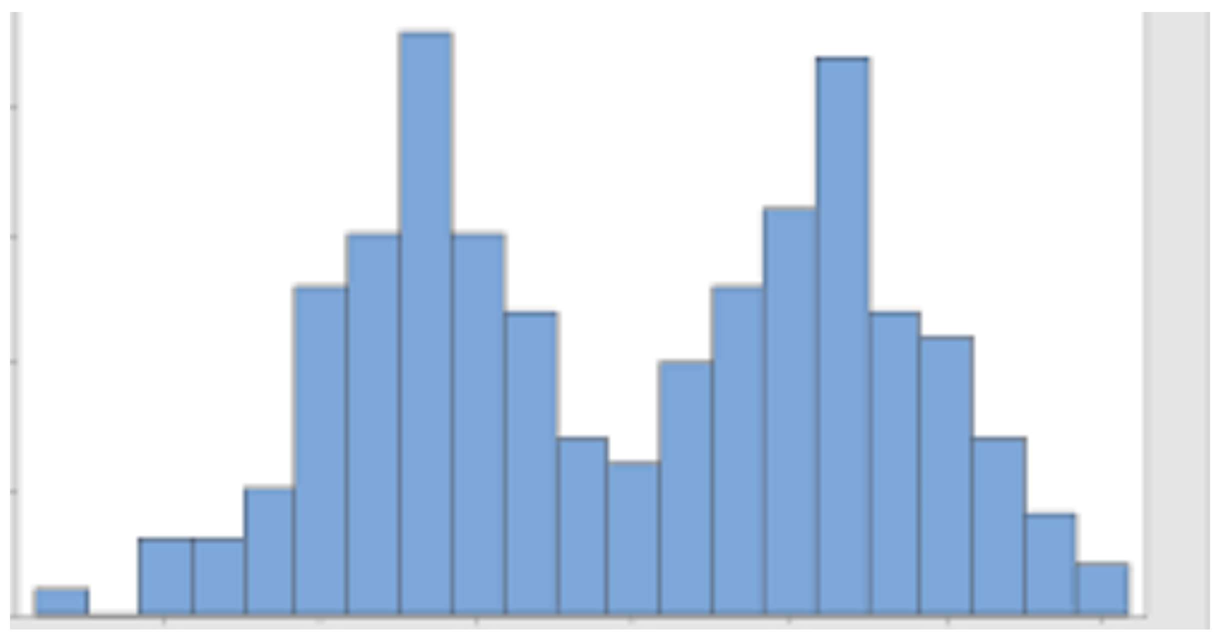
p(y) for X = 1, nonfiction books

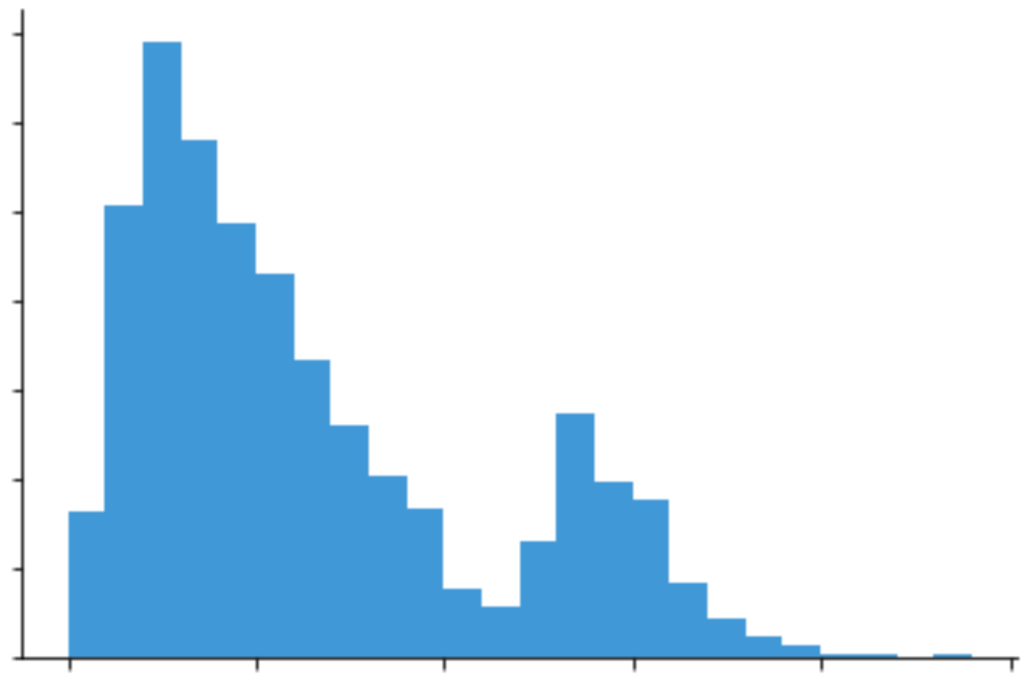


A long tail, a few very long books

Conditional Expectation Example (cont)

- What do we mean by $p(y \mid X = 0)$? How might it differ from $p(y \mid X = 1)$





- $\mathbb{E}[Y \mid X = 0]$ is the expectation over Y under distribution $p(y \mid X = 0)$
- $\mathbb{E}[Y \mid X = 1]$ is the expectation over Y under distribution $p(y \mid X = 1)$

Conditional Expectations

Definition:
The **expected value of Y conditional on $X = x$** is

$$\mathbb{E}[Y \mid X = x] = \begin{cases} \sum_{y \in \mathcal{Y}} y p(y \mid x) & \text{if } Y \text{ is discrete,} \\ \int_{\mathcal{Y}} y p(y \mid x)\, dy & \text{if } Y \text{ is continuous.} \end{cases}$$

Question: What is $\mathbb{E}[Y \mid X]$?

Conditional Expectations

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The **expected value of Y conditional on $X = x$** is

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Question: What is $\mathbb{E}[Y \mid X]$?

Answer: $Z = \mathbb{E}[Y \mid X]$ is a random variable, $z = \mathbb{E}[Y \mid X = x]$ is an outcome

Properties of Expectations

- Linearity of expectation:
  - $\mathbb{E}[cX] = c\mathbb{E}[X]$ for all constant c
  - $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Products of expectations of **independent** random variables X, Y:
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Law of Total Expectation:
  - $\mathbb{E}\left[\mathbb{E}\left[Y \mid X\right]\right] = \mathbb{E}[Y]$
- **Question:** How would you prove these?

Linearity of Expectation

$$\mathbb{E}[X+Y] = \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)(x+y)$$
$$= \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)(x+y)$$
$$= \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)x + \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)y$$

$$\sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)x = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)x$$
$$= \sum_{x\in\mathcal{X}} x \sum_{y\in\mathcal{Y}} p(x,y) \qquad \triangleright\ p(x) = \sum_{y\in\mathcal{Y}} p(x,y)$$
$$= \sum_{x\in\mathcal{X}} xp(x)$$
$$= \mathbb{E}[X]$$

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$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)x = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)x$$
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$$= \sum_{x\in\mathcal{X}} xp(x)$$
$$= \mathbb{E}[X]$$

What if the RVs are continuous?

$$\mathbb{E}[X+Y] = \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)(x+y)$$
$$= \sum_{y\in\mathcal{Y}} \sum_{x\in\mathcal{X}} p(x,y)(x+y)$$
$$= \sum_{y\in\mathcal{Y}} \sum_{x\in\mathcal{X}} p(x,y)x + \sum_{y\in\mathcal{Y}} \sum_{x\in\mathcal{X}} p(x,y)y$$
$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\mathbb{E}[X+Y] = \int_{\mathcal{X}\times\mathcal{Y}} p(x,y)(x+y)d(x,y)$$
$$= \int_{\mathcal{Y}} \int_{\mathcal{X}} p(x,y)(x+y)dxdy$$
$$= \int_{\mathcal{Y}} \int_{\mathcal{X}} p(x,y)xdxdy + \int_{\mathcal{Y}} \int_{\mathcal{X}} p(x,y)ydxdy$$
$$= \int_{\mathcal{X}} x \int_{\mathcal{Y}} p(x,y)dydx + \int_{\mathcal{Y}} y \int_{\mathcal{X}} p(x,y)dxdy$$
$$= \int_{\mathcal{X}} xp(x)dx + \int_{\mathcal{Y}} yp(y)dy$$
$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

Properties of Expectations

- Linearity of expectation:
  - $\mathbb{E}[cX] = c\mathbb{E}[X]$ for all constant c
  - $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Products of expectations of **independent** random variables X, Y:
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Law of Total Expectation:
  - $\mathbb{E}\left[\mathbb{E}\left[Y \mid X\right]\right] = \mathbb{E}[Y]$
- **Question:** How would you prove these?

$$\mathbb{E}[Y] = \sum_{y \in \mathcal{Y}} y p(y) \quad \text{def. E[Y]}$$

$$= \sum_{y \in \mathcal{Y}} y \sum_{x \in \mathcal{X}} p(x, y) \quad \text{def. marginal distribution}$$

$$= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} y p(x, y) \quad \text{rearrange sums}$$

$$= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} y p(y \mid x) p(x) \quad \text{Chain rule}$$

$$= \sum_{x \in \mathcal{X}} \left(\sum_{y \in \mathcal{Y}} y p(y \mid x) \right) p(x)$$

$$= \sum_{x \in \mathcal{X}} \left(\mathbb{E}[Y \mid X = x] \right) p(x) \quad \text{def. E[Y | X = x]}$$

$$= \sum_{x \in \mathcal{X}} \left(\mathbb{E}[Y \mid X = x] \right) p(x)$$

$$= \mathbb{E}\left(\mathbb{E}[Y \mid X] \right) \blacksquare \quad \text{def. expected value of function}$$

Variance

Definition: The **variance** of a random variable is

$$\text{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right].$$

i.e., $\mathbb{E}[f(X)]$ where $f(x) = (x - \mathbb{E}[X])^2$.

Equivalently,

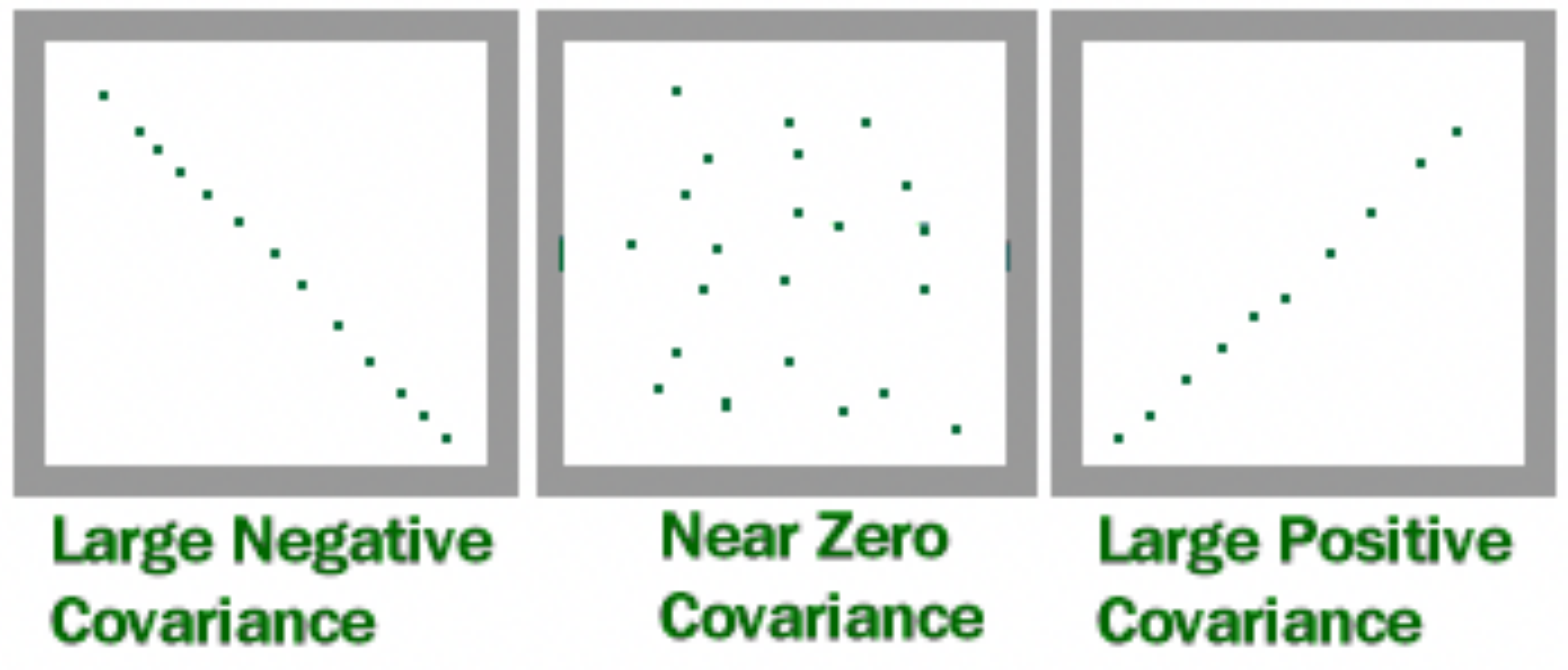
$$\text{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$$

(**why?**)

Covariance

Definition: The covariance of two random variables is

$$\text{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$
$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\,.$$

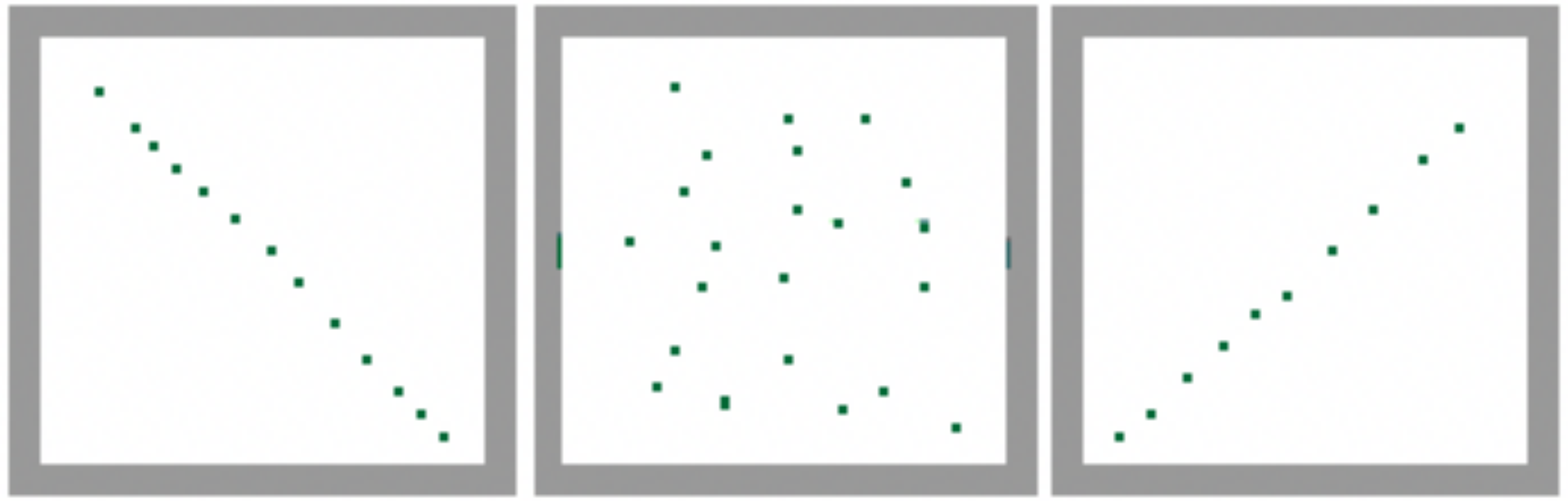


Question: What is the range of $\text{Cov}(X, Y)$?

Correlation

Definition: The **correlation** of two random variables is

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$



Question: What is the range of $\text{Corr}(X, Y)$?

hint: $\text{Var}(X) = \text{Cov}(X, X)$

Properties of Variances

- Var$[c] = 0$ for constant c
- Var$[cX] = c^2$Var$[X]$ for constant c
- Var$[X + Y] =$ Var$[X] +$ Var$[Y] + 2$Cov$[X, Y]$
- For **independent** X, Y,
  Var$[X + Y] =$ Var$[X] +$ Var$[Y]$ (**why?**)

Independence and Decorrelation

- Independent RVs have zero correlation (**why?**)

  hint: Cov$[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

- Uncorrelated RVs (i.e., Cov$(X, Y) = 0$) **might be dependent** (i.e., $p(x, y) \neq p(x)p(y)$).
  - Correlation (**Pearson's correlation coefficient**) shows linear relationships; but can miss nonlinear relationships
  - **Example:** $X \sim$ Uniform$\{-2, -1, 0, 1, 2\}$, $Y = X^2$
    - $\mathbb{E}[XY] = .2(-2 \times 4) + .2(2 \times 4) + .2(-1 \times 1) + .2(1 \times 1) + .2(0 \times 0)$
    - $\mathbb{E}[X] = 0$
    - So $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0 - 0\mathbb{E}[Y] = 0$

Summary

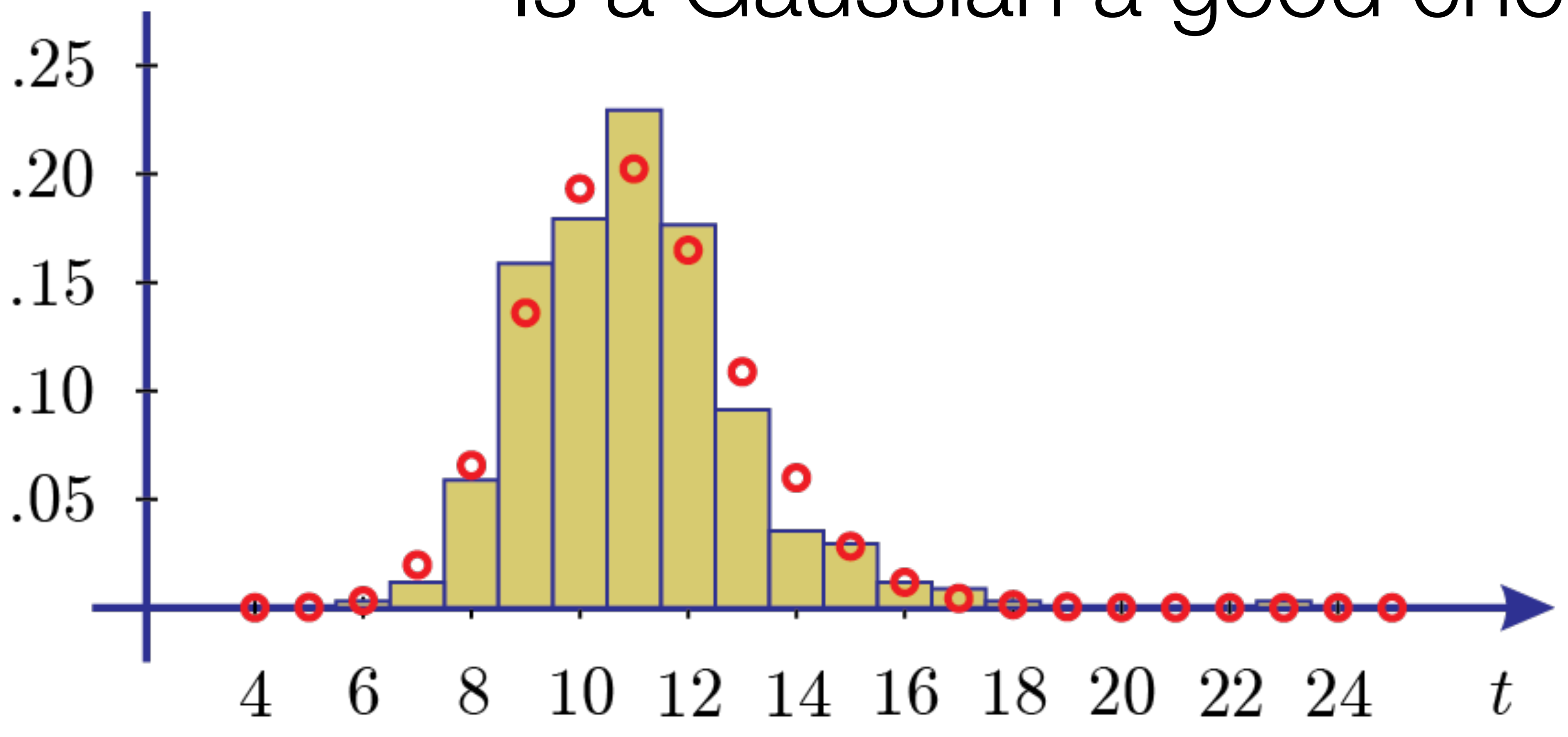
- **Random variables** are functions from sample to some value
  - Upshot: A random variable takes different values with some probability
- The value of one variable can be informative about the value of another (because they are both functions of the same sample)
  - Distributions of multiple random variables are described by the **joint** probability distribution (joint PMF or joint PDF)
  - You can have a new distribution over one variable when you **condition** on the other
- The **expected value** of a random variable is an **average** over its values, **weighted** by the probability of each value
- The **variance** of a random variable is the expected squared distance from the mean
- The **covariance** and **correlation** of two random variables can summarize how changes in one are informative about changes in the other.

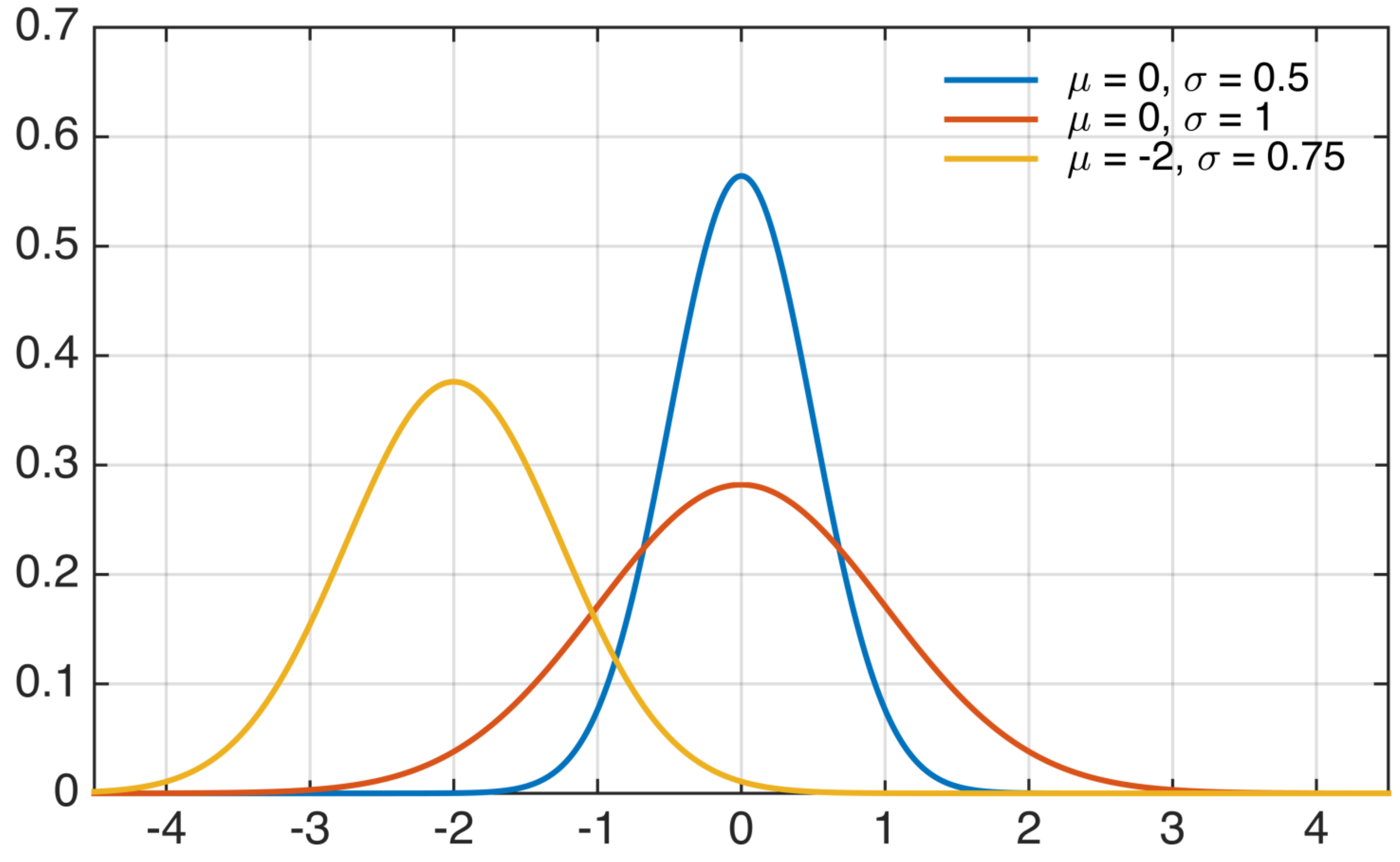
Exercise applying your knowledge

- Let's revisit the commuting example, and assume we collect continuous commute times
- We want to model commute time as a Gaussian

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

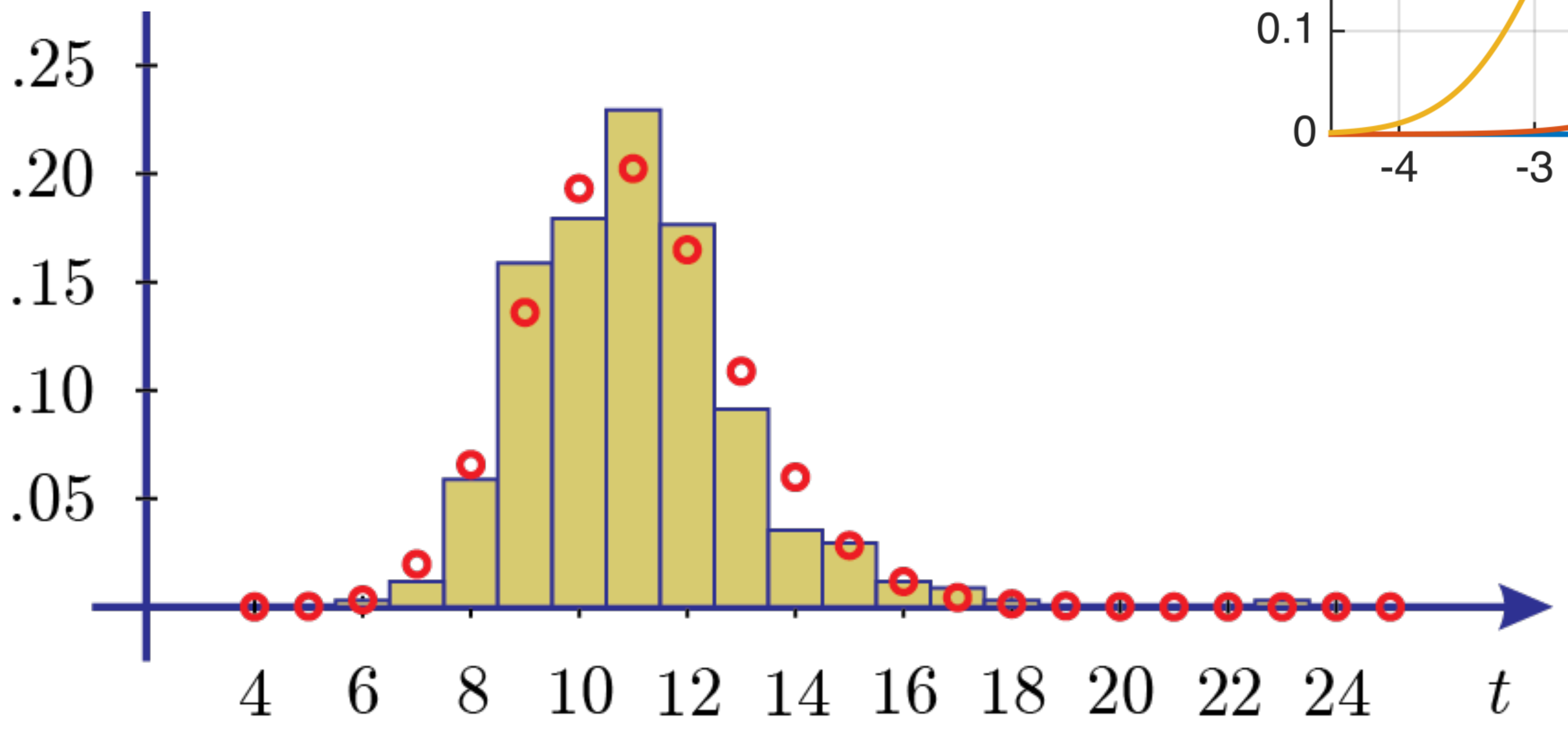
- What parameters do I have to specify (or learn) to model commute times with a Gaussian?
- Is a Gaussian a good choice?

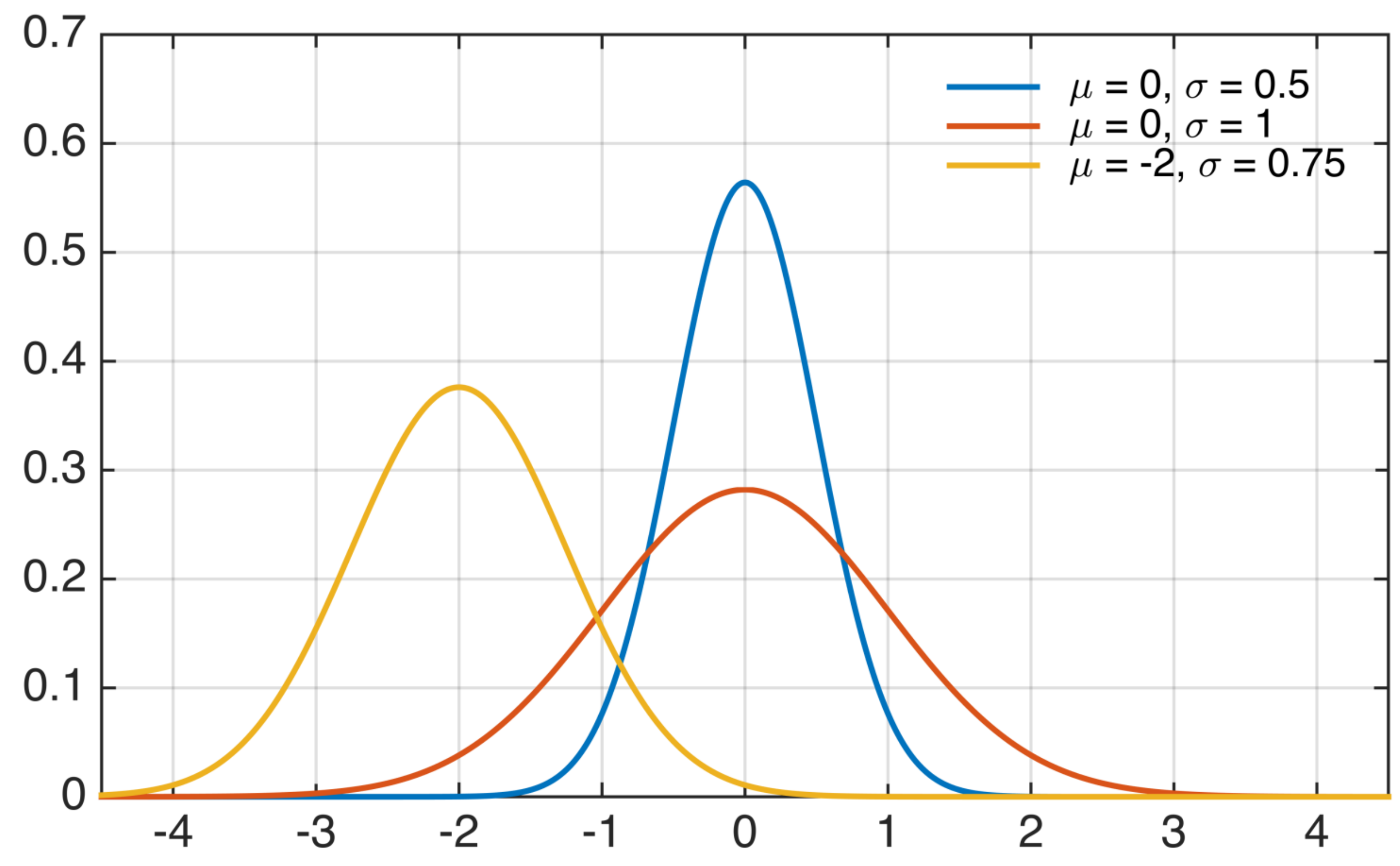


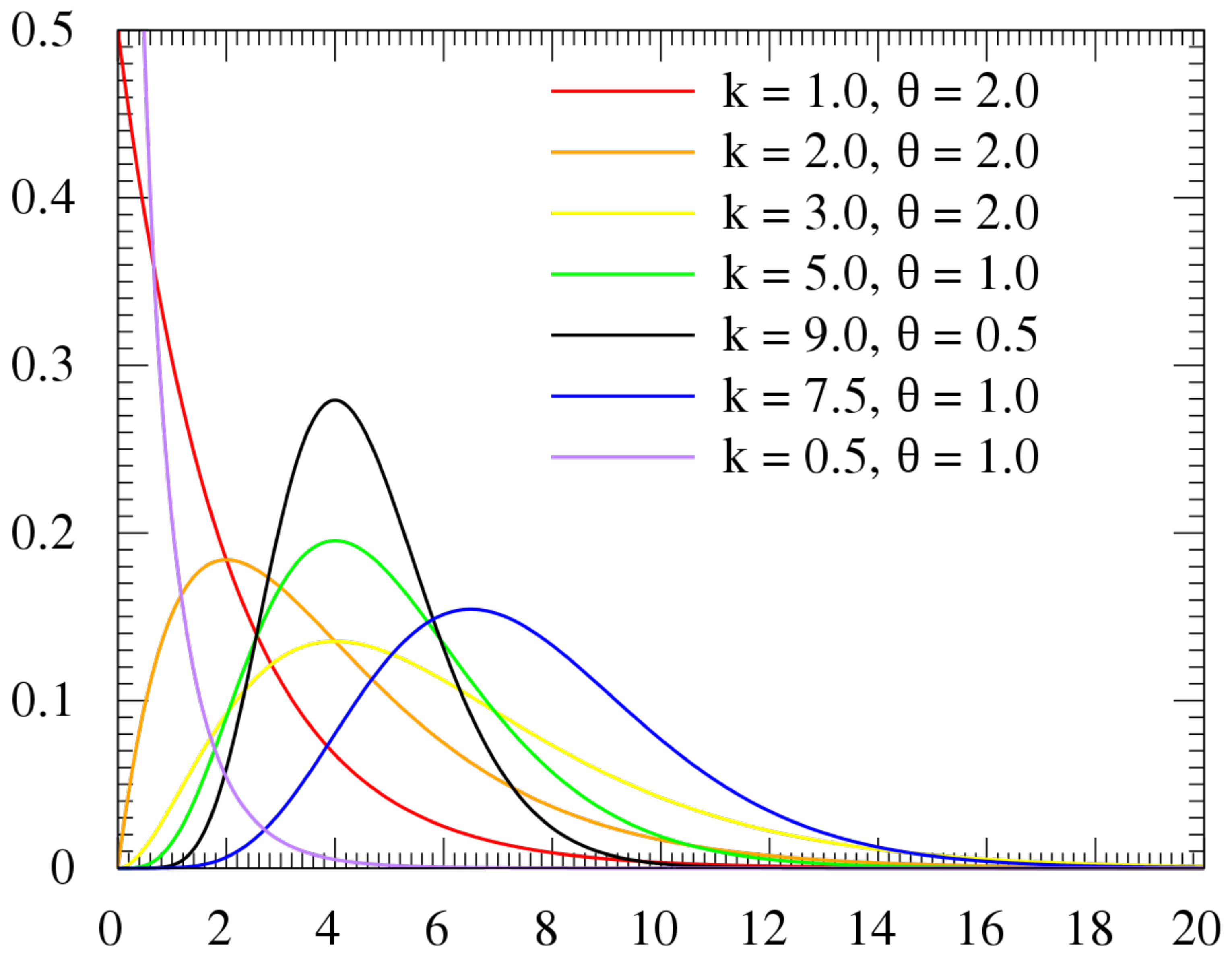


Exercise applying your knowledge

- A better choice is actually what is called a Gamma distribution

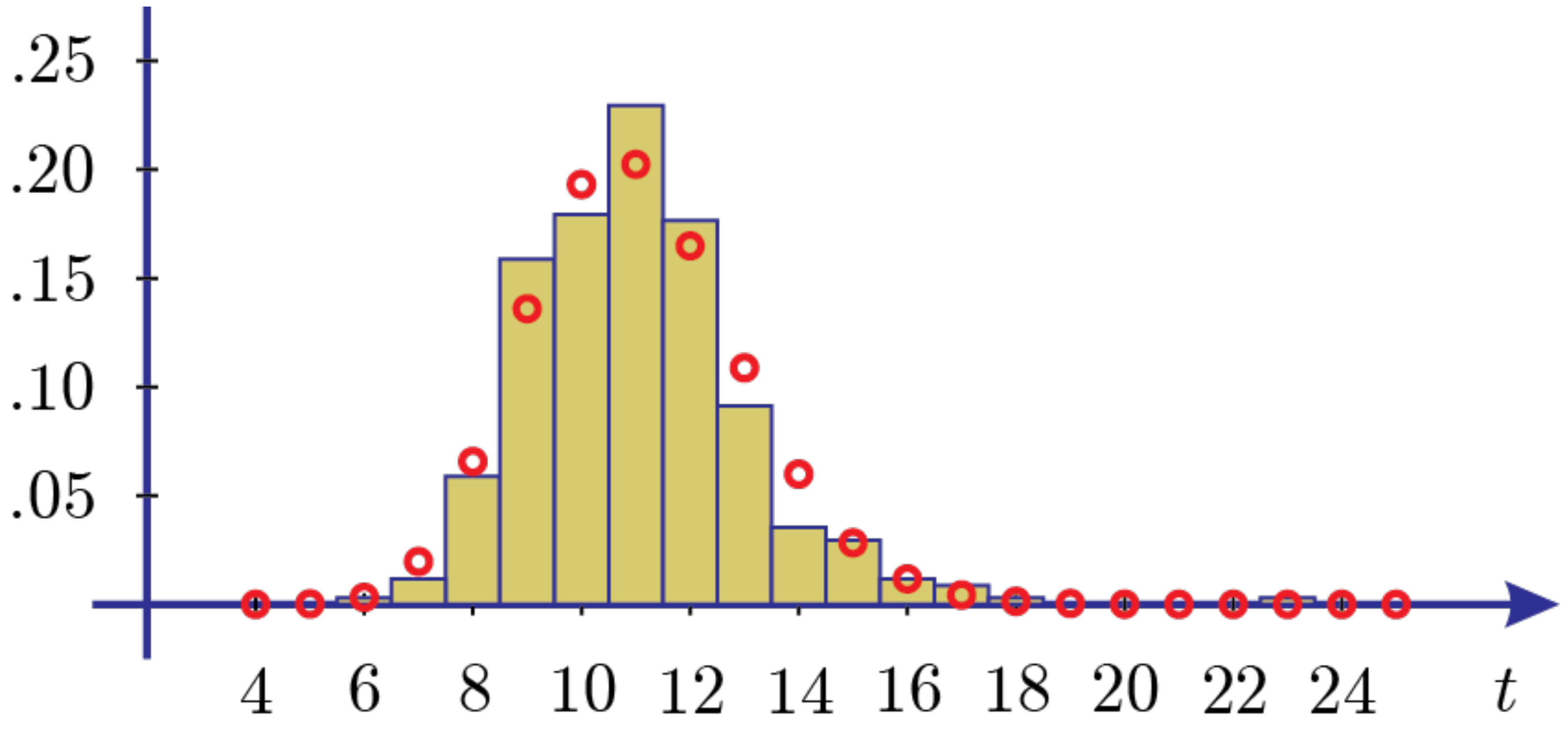






Exercise applying your knowledge

- We can also consider conditional distributions $p(y \mid x)$
- Y is the commute time, let X be the month
- Why is it useful to know $p(y \mid X = \text{Feb})$ and $p(y \mid X = \text{Sept})$?
- What else could we use for X and why pick it?



Exercise applying your knowledge

- Let's use a simple X, where it is 1 if it is slippery out and 0 otherwise
- Then we could model two Gaussians, one for the two types of conditions

$$p(y|X=0) = \mathcal{N}\left(\mu_0, \sigma_0^2\right)$$
$$p(y|X=1) = \mathcal{N}\left(\mu_1, \sigma_1^2\right)$$

Gaussian denoted by N